Deciding Hopf Bifurcations by Quantifier Elimination in a Software-component Architecture

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In this paper we give a semi-algebraic description of Hopf bifurcation fixed points for a given parameterized polynomial vector field. The description is carried out by use of the Hurwitz determinants, and produces a first-order formula which is transformed into a quantifier-free formula by the use of usual-quantifier elimination algorithms. We apply techniques from the theory of sub-resultant sequences and of Gröbner bases to come up with efficient reductions, which lead to quantifier elimination questions that can often be handled by existing quantifier elimination packages.

We could implement the algorithms for the conditions on Hopf bifurcations by combining the computer algebra system Maple with packages for quantifier elimination using a Java-based component architecture recently developed by the second author. In addition to some textbook examples we applied our software system to an example discussed in a recent research paper.

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1. Introduction

Systems of ordinary differential equations are one of the most common mathematical structures used to model processes in the natural sciences. In general, these models lead to nonlinear systems which depend on parameters. Depending on the parameters their behavior might change dramatically. Thus, the development of symbolic methods for their study is an important topic.

During the last decade many advances have been made for the symbolic study of differential equations (Singer, 1990a; Seiler, 1997) using techniques such as Lie-symmetry methods (Olver, 1986; Stephani, 1989; Hereman, 1994) or differential Galois theory (Singer, 1990b; Singer and Ulmer, 1993). However, this work is mainly aimed towards the symbolic solutions of systems and as many important examples are not solvable in symbolic form, these techniques are not applicable.

Nevertheless, very often only the qualitative behavior of a system of ordinary differential equations in dependency on the parameters is of interest. In this respect great advances have been obtained recently, e.g. for various questions of stability, such as the ones for...
certain numerical integration schemes (Hong et al., 1997) or in connection with control theory (Jirstrand, 1997).

That work uses the powerful technique of quantifier elimination on real closed fields (Tarski, 1951), to which the questions on the differential equations are reduced—in the common case that the corresponding vector field is a polynomial system in the variables and parameters.

In this paper we are concerned with a question that is similar in spirit to those results but in the context of a question that has not been treated within the present context before: is it possible to reduce the various natural questions in connection with the existence of Hopf bifurcations on parameterized polynomial vector fields to quantifier elimination problems? The main result of this paper is a positive answer to this question. We will give a semi-algebraic description of Hopf bifurcation fixed points for a given parameterized polynomial vector field. In order to obtain this result we first establish a link between Hurwitz determinants and principal sub-resultant coefficients, and then we study the behavior of the Hurwitz determinants in the presence of symmetric roots. Using these techniques along with some simplifications via Gröbner basis techniques we obtain quite efficient reductions to quantifier elimination problems, which allow us to compute several examples.

Our work on using symbolic methods on Hopf bifurcations can be seen as being orthogonal to the one using perturbation methods, see e.g. Rand and Armbruster (1987). These methods are not restricted to polynomial vector fields, but are restricted to ones which have the form of perturbated systems, whereas our method works for all polynomial vector fields.

A prerequisite of our work was a system infrastructure for symbolic software components (Weber et al., 1998), which allowed us to connect various existing quantifier elimination packages with a general-purpose computer algebra system, which was used to perform the reductions to quantifier elimination problems.

1.1. Outline of the paper

The necessary preliminaries on ordinary differential equations will be given in Section 2. The result on the semi-algebraic description of Hopf bifurcation fixed points is developed in Section 3. The Java-based software-component infrastructure that was necessary to implement the algorithms is described in Section 4. Computational examples are given in Section 5.

2. Preliminaries

For ordinary differential equations a basic question is to ask whether a given system has a unique solution that depends continuously on its parameters and initial values, i.e. the well posedness.

When a given problem is well posed, then the next most important questions concern the stability of its solutions. Roughly speaking, a solution is stable if any solution starting out close to the given solution remains close to it. When the solutions starting out close to the given solution become arbitrary close to it, then the solution is said to be asymptotically stable.

Even if a given nonlinear system $\dot{x} = f(x)$ is finite dimensional, the stability study of its solutions is a very hard problem in its general setting. To overcome this difficulty one
can restrict the study to the stability of a specified class of solutions. The first class of solutions one can take as a starting point to study the system is the class of constant solutions, also called fixed points, which are the solutions of the equation

\[ f(x) = 0. \]

For a given fixed point \( \bar{x} \) of a \( C^\infty \) vector field \( f \) the study of the system near this point is classically done by Taylor expanding \( f \) near \( \bar{x} \), and by first considering the linear system

\[ \dot{\zeta} = Df(\bar{x}) \cdot \zeta, \]

where \( Df(\bar{x}) \) is the Jacobian matrix of \( f \) at the point \( \bar{x} \).

When the matrix \( Df(\bar{x}) \) has no eigenvalue with zero real part, then the stability study of the nonlinear system near the point \( \bar{x} \) reduces to the study of the stability of the linear system near the origin 0. In the presence of eigenvalues with zero real part, the linear system gives only partial information about the local dynamics of the nonlinear system near the point \( \bar{x} \). In fact, the local behavior near \( \bar{x} \) of the nonlinear system depends on the higher order terms of the Taylor expansion of \( f \) near the point \( \bar{x} \). However, the number of eigenvalues with zero real part of \( Df(\bar{x}) \) is a fundamental invariant in the study of the topological nature of the local dynamics near the point \( \bar{x} \).

2.1. Testing Stability

Let \( K \) be a discrete subfield of the real numbers field \( \mathbb{R} \) and let \( f(u, x) = (f_1, \ldots, f_n) \) be a parameterized vector field, where \( f_i \in K[u, x] \) are polynomials of degree \( \leq d \), \( x = (x_1, \ldots, x_n) \) is a list of variables and \( u = (u_1, \ldots, u_k) \) is a list of parameters. Let us consider the autonomous ordinary differential system

\[ \dot{x} = f(u, x) \]

and let us denote by \( \Phi_t(u, x) \) the flow generated by the vector field \( f \).

A good place to start the study of the nonlinear system \( \dot{x} = f(u, x) \) is to find its fixed points, also called equilibria, which are given by the equation

\[ f(u, x) = 0. \]

If the list of parameters \( u \) is given a value \( \bar{u} \in \mathbb{R}^k \), and \( (\bar{u}, \bar{x}) \) is a fixed point of the specialized nonlinear system \( \dot{x} = f(\bar{u}, x) \), the study of the behavior of the flow \( \Phi_t(\bar{u}, x) \) when starting near the fixed point \( (\bar{u}, \bar{x}) \) is classically done by using the linear system

\[ \dot{\zeta} = Df(\bar{u}, \bar{x}) \cdot \zeta, \]

where \( Df(\bar{u}, \bar{x}) \) is the Jacobian matrix of the vector field \( f(\bar{u}, x) \) at the point \( \bar{x} \). The flow generated by this linear system is then \( e^{tDf(\bar{u}, \bar{x})} \cdot \zeta = D\Phi_t(\bar{u}, \bar{x}) \cdot \zeta \).

2.2. Hyperbolic Fixed Points

A fundamental result due to Hartman and Grobman (see Arnold, 1973) states that in the case of a hyperbolic fixed point, i.e. the matrix \( Df(\bar{u}, \bar{x}) \) has no eigenvalue with zero real part, the nonlinear flow has the same behavior near the fixed point \( (\bar{u}, \bar{x}) \) as the linear flow near the origin 0. In particular, the nonlinear flow \( \Phi_t(u, x) \) is asymptotically stable near the fixed point \( (\bar{u}, \bar{x}) \) if and only if all the eigenvalues of the matrix \( Df(\bar{u}, \bar{x}) \) have negative real part.
According to the well known Routh–Hurwitz criterion, see e.g. Hong et al. (1997) and also Section 3, this last condition is equivalent to the signs conjunction

$$\Delta_1(u, x) > 0, \ldots, \Delta_n(u, x) > 0,$$

where the $\Delta_i(u, x)$'s are the Hurwitz determinants associated to the characteristic polynomial of the matrix $Df(u, x)$.

As the nonlinear system $\dot{x} = f(u, x)$ is parameterized, a natural question is to ask for which values $u$ of the parameter $u$ the specialized system $\dot{x} = f(u, x)$ is asymptotically stable near all its fixed points. This can be symbolically expressed by the first-order formula

$$\forall x (f(u, x) = 0 \Rightarrow \Delta_1(u, x) > 0, \ldots, \Delta_n(u, x) > 0).$$

One can also ask for which values $u$ of the parameter $u$ the specialized system $\dot{x} = f(u, x)$ is asymptotically stable near at least one of its fixed points. That is

$$\exists x (f(u, x) = 0, \Delta_1(u, x) > 0, \ldots, \Delta_n(u, x) > 0).$$

These questions, as many others, are thus reduced to quantifier elimination problems for first-order formulas in the language of real closed fields.

2.3. bifurcations

When the matrix $Df(u, x)$ has some eigenvalues with zero real part, that is, the system undergoes a bifurcation, and for $(u, x)$ close enough to the fixed point $(u, x)$, radically new dynamical behavior can occur. For example, fixed points can be created or destroyed, and even new orbits such as periodic or quasiperiodic ones can be created.

In general, for an $n$-dimensional autonomous system there are many distinct bifurcating situations, and their systematic study is a hard problem. For this reason we restrict ourselves to the study of a specified nonhyperbolic situation, namely the Hopf bifurcation.

In this situation the matrix $Df(u, x)$ has two distinct imaginary eigenvalues and no other eigenvalue with zero real part. When the system undergoes such bifurcation at a fixed point $(u, x)$, and the parameters $u$ are subjected to small perturbations, the original equilibrium point $(u, x)$ moves analytically in terms of $u$ and no new equilibrium is created in the neighborhood. However, if the imaginary eigenvalues of the linearized system move away from the imaginary axis, one expects the equilibrium point to change its stability type. This change is typically marked by the appearance of a small periodic orbit encircling the equilibrium point as stated by the Poincaré–Andronov–Hopf theorem, see e.g. Chow and Hale (1996). The local dynamics near an equilibrium point with Hopf bifurcation cannot be determined by the linear approximation of the vector field. In fact, depending on the nonlinear terms of $f$, the fixed point can be unstable, stable or even asymptotically stable.

In the following we shall not consider stability questions for the Hopf bifurcation fixed points. Our attention will be focused on giving necessary and sufficient conditions on the parameters for the vector field to undergo a Hopf bifurcation. We shall prove that this problem can be expressed by a first-order formula in the language of ordered fields and hence turns out to be a quantifier elimination problem.

The results of this section are valid for arbitrary real closed fields. Let \( \mathbb{R} \) be a real closed field and \( \mathbb{C} = \mathbb{R}(i) \) its algebraic closure. Let \( \chi(z) \in \mathbb{R}[z] \) be a polynomial of degree \( n \), (typically \( \chi(z) \) is the characteristic polynomial of a square matrix of order \( n \)), and let us write
\[
\chi(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n.
\]
The square matrix of order \( n \)
\[
H = \begin{pmatrix}
a_1 & a_3 & a_5 & \cdots & a_n \\
a_0 & a_2 & a_4 & \cdots & a_n \\
0 & a_1 & a_3 & a_5 & \cdots \\
0 & a_0 & a_2 & a_4 & \cdots \\
\ddots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

is called the Hurwitz matrix of the polynomial \( \chi(z) \). The \( i \)th order principal minor of the matrix \( H \) is called the \( i \)th Hurwitz determinant of the matrix \( H \) and is denoted by \( \Delta_i \). The well known Routh–Hurwitz criterion states that the polynomial \( \chi \) has all its roots in the left half-plane if and only if its Hurwitz determinants verify the sign conditions
\[
\Delta_1 > 0, \ldots, \Delta_n > 0.
\]

3.1. Hurwitz determinants and sub-resultant sequences

In the following we shall use the Hurwitz determinants to give a similar criterion for the polynomial \( \chi \) to have \( k \) pairs of symmetric roots with respect to the origin of the plane. We shall also give a criterion for the polynomial \( \chi \) to have all its roots in the left half-plane except two roots \( i \omega \) and \( -i \omega \) which are on the imaginary axis. For this, we shall first express the Hurwitz determinants in terms of the principal sub-resultant coefficients of a pair of polynomials which are related to the polynomial \( \chi(z) \).

Let \( \chi(z) \in \mathbb{R}[z] \) be a polynomial of degree \( n \),
\[
\chi(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n.
\]
Let us decompose the polynomial \( \chi \) into the form
\[
\chi(z) = \chi_1(z^2) + z \chi_2(z^2).
\]
If \( n \) is even, \( n = 2m \), then
\[
\chi_1 = a_0 y^m + a_2 y^{m-1} + \cdots + a_n, \quad \chi_2 = a_1 y^{m-1} + a_3 y^{m-2} + \cdots + a_{n-1}.
\]
In this case \( H \) is the matrix having as rows the coordinates of the vectors
\[
y^m \chi_2(y), y^{m-1} \chi_1(y), \ldots, y \chi_2(y), y \chi_1(y), \ldots, \chi_2(y), \chi_1(y)
\]
in the canonical basis \( \{ y^{n-1}, \ldots, y, 1 \} \) of the vector space \( \mathbb{R}_{n-1}[y] \).

On the other hand, if \( n \) is odd, \( n = 2m + 1 \), then
\[
\chi_1 = a_1 y^n + a_3 y^{n-1} + \cdots + a_n, \quad \chi_2 = a_0 y^n + a_2 y^{n-1} + \cdots + a_{n-1}.
\]
In this case \( H \) is the matrix having as rows the coordinates of the vectors
\[
y^m \chi_1(y), y^n \chi_2(y), \ldots, y \chi_1(y), y \chi_2(y), \ldots, y \chi_1(y), \chi_2(y), \chi_1(y)
\]
in the canonical basis \( \{ y^{n-1}, \ldots, y, 1 \} \) of the vector space \( \mathbb{R}_{n-1}[y] \).
Definition 3.1. Let $P, Q \in \mathbb{R}[y]$ be two polynomials,

$$P = \sum_{k=0}^{p} a_k y^k \quad Q = \sum_{k=0}^{q} b_k y^k$$

with $\deg(P) \leq p$ and $\deg(Q) \leq q$.

If $i \in \{0, \ldots, \inf(p, q) - 1\}$ we define the sub-resultant polynomial associated to $P, p$ and $Q, q$ of index $i$ as follows:

$$Sr_i(P, p, Q, q) = \sum_{j=0}^{i} d_{ij} y^j,$$

where every $d_{ij}$ is the determinant of the matrix built with the columns $1, 2, \ldots, p+q-2i-1$ and $p+q-i-j$ in the following matrix:

$$M_i = \begin{pmatrix}
    a_p & \ldots & a_0 \\
    \vdots & \ddots & \vdots \\
    b_q & \ldots & b_0 \\
    \vdots & \ddots & \vdots \\
    b_q & \ldots & b_0
\end{pmatrix}
\begin{cases}
    \text{q - i} & \\
    \text{p - i}
\end{cases}

The determinant $d_{ij}$ is called the $i$th principal sub-resultant coefficient and is denoted by $sr_i(P, p, Q, q)$.

When no confusion arises, we shall write $sr_i$ and $Sr_i$ for $sr_i(P, p, Q, q)$ and $Sr_i(P, p, Q, q)$.

The Hurwitz determinants sequence of the polynomial $\chi(z)$ is in fact closely related to the principal sub-resultant sequence of the pair of polynomials $\chi_2$ and $\chi_1$.

More precisely, one has the following result.

Theorem 3.1. Let $\chi \in \mathbb{R}[z]$ be a polynomial of degree $n$, and write

$$\chi(z) = \chi_1(z^2) + z\chi_2(z^2).$$

Let $\Delta_1, \Delta_2, \ldots$ be the Hurwitz determinants sequence of $\chi$ and $sr_0, sr_1, \ldots$ be the principal sub-resultant coefficients sequence of the polynomials $\chi_2$ and $\chi_1$ (with $\deg(\chi_1) \leq \lfloor n/2 \rfloor$ and $\deg(\chi_2) \leq \lfloor (n-1)/2 \rfloor$), then

$$\Delta_{n-2i-1} = \epsilon_i sr_i,$$

where $\epsilon_i = (-1)^{(m-i)(m-i-1)/2}$ if $n = 2m$ and $\epsilon_i = (-1)^{(m-i)(m-i+1)/2}$ if $n = 2m + 1$.

Proof. Let us suppose first that $n = 2m$. The Hurwitz determinant $\Delta_{n-2i-1}$ is then the principal minor of the matrix built with the rows

$$y^m \chi_2, \ y^{m-1} \chi_1, \ldots, y^{i+2} \chi_2, \ y^{i+1} \chi_1, \ y^i \chi_2.$$

On the other hand, the principal sub-resultant coefficient $sr_i$ is the principal minor of the matrix built with the rows

$$y^{m-1} \chi_2, \ y^{m-2} \chi_2, \ldots, y^i \chi_2, \ y^{m-2} \chi_1, \ y^{m-3} \chi_1, \ldots, y^i \chi_1.$$
which is also the principal minor of the matrix having as rows
\[ y^n \chi_2, y^{n-1} \chi_2, \ldots, y^1 \chi_2, y^{n-1} \chi_1, y^{n-2} \chi_1, \ldots, y^1 \chi_1. \]

Let us note that this last matrix can be obtained from the first one by exchanging rows according to the permutation \( \sigma \in S_{n-2i-1} \) defined by
\[
\sigma(j) = \begin{cases} 
    j - k & \text{if } j = 2k + 1 \\
    k + m - i & \text{if } j = 2k.
\end{cases}
\]

We thus have \( \Delta_{n-2i-1} = \epsilon(\sigma) \text{sr}_i(\chi_2, \chi_1). \]

As consequence of Theorem 3.1 we have the following.

**Corollary 3.2.** Let \( \chi(z) \in \mathbb{R}[z] \) be a polynomial of degree \( n \). Then \( \chi \) has \( k \) pairs of symmetric roots \( z_j \) and \( -z_j \) if and only if
\[
\Delta_{n-1} = 0, \ldots, \Delta_{n-2k+1} = 0, \quad \Delta_{n-2k-1} \neq 0.
\]

**Proof.** The number of symmetric roots, counted with multiplicities, of the polynomial \( \chi \) is equal to the number of common roots, counted with multiplicities, of the two polynomials \( \chi_1 \) and \( \chi_2 \). According to elementary properties of sub-resultant sequences the polynomials \( \chi_1 \) and \( \chi_2 \) have \( k \) common roots if and only if
\[
\text{sr}_0 = 0, \ldots, \text{sr}_{k-1} = 0, \quad \text{sr}_k \neq 0,
\]
which is equivalent to
\[
\Delta_{n-1} = 0, \ldots, \Delta_{n-2k+1} = 0, \quad \Delta_{n-2k-1} \neq 0
\]
according to Theorem 3.1. □

**Remark 3.1.** Let \( z_1, \ldots, z_n \) be the complex roots of the polynomial \( \chi(z) \). Then using Orlando’s formula, see Gantmacher (1959), one obtains
\[
\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_0^{-1} \prod_{i<j}(z_i + z_j),
\]
and hence, the determinant \( \Delta_{n-1} \) vanishes if and only if the polynomial \( \chi(z) \) has at least one pair of symmetric roots with respect to the origin. In contrast to Corollary 3.2, Orlando’s formula gives no information on the number of symmetric pairs which the polynomial has.

### 3.2. Hurwitz determinants in the case of symmetric roots

We now turn our investigation to another aspect of the Hurwitz determinants, namely the behavior of the \( \Delta_i \)'s when we add to the roots of the polynomial \( \chi \) some pairs of symmetric points \( z_j \) and \( -z_j \) of the plane.

**Lemma 3.3.** Let \( \chi(z) \in \mathbb{R}[z] \) be a polynomial of degree \( n \), \( z_1 \) be an arbitrary complex
number and $\chi^*(z) = \chi(z)(z^2 - z^2_1)$. If $\Delta^*_i$ is the Hurwitz determinant of order $i$ of the polynomial $\chi^*(z)$, then

$$\Delta_i = \Delta^*_i \quad \text{for} \quad i = 1, \ldots, n.$$ 

**Proof.** For $i \leq n$ the Hurwitz determinant $\Delta^*_i$ is the determinant of the square order $i$ matrix

$$H^*_i = \begin{pmatrix}
    a_1 & a_3 - z_1^2 a_1 & a_5 - z_1^2 a_3 & \cdots & . \\
    a_0 & a_2 - z_1^2 a_0 & a_4 - z_1^2 a_2 & \cdots & . \\
    0 & a_1 & a_3 - z_1^2 a_1 & \cdots & . \\
    0 & a_0 & a_2 - z_1^2 a_0 & \cdots & . \\
    . & . & . & \ddots & . \\
    . & . & . & \ddots & \ddots \\
    . & . & . & \ddots & \ddots \\
    0 & a_1 & a_3 & \cdots & a_i - z_1^2 a_{i-2}
\end{pmatrix}.$$ 

On the other hand, the Hurwitz determinant $\Delta_i$ is the determinant of the square order $i$ matrix

$$H_i = \begin{pmatrix}
    a_1 & a_3 & a_5 & \cdots & . \\
    a_0 & a_2 & a_4 & \cdots & . \\
    0 & a_1 & a_3 & \cdots & . \\
    0 & a_0 & a_2 & \cdots & . \\
    . & . & . & \ddots & . \\
    . & . & . & \ddots & \ddots \\
    . & . & . & \ddots & \ddots \\
    . & . & . & \ddots & \ddots \\
    0 & a_1 & a_3 & \cdots & a_i 
\end{pmatrix},$$

since $H^*_i = H_i A$, where

$$A = \begin{pmatrix}
    1 & -z_1^2 & \cdots & \cdots & \cdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    1 & \cdots & \cdots & z_1^2 & 1
\end{pmatrix}.$$ 

Then the two matrices have the same determinant, i.e. $\Delta_i = \Delta^*_i$. \(\square\)

Using induction on the number of added pairs we obtain the following proposition.

**Proposition 3.4.** Let $\chi(z) \in \mathbb{R}[z]$ be a polynomial of degree $n$ and $z_1, \ldots, z_k$ be arbitrary complex numbers. Let

$$\chi^*(z) = \chi(z)(z^2 - z^2_1) \cdots (z^2 - z^2_k).$$

If $\Delta^*_i$ is the Hurwitz determinant of order $i$ of the polynomial $\chi^*(z)$, then

$$\Delta_i = \Delta^*_i \quad \text{for} \quad i = 1, \ldots, n.$$ 

**Remark 3.2.** If we add to the roots of the polynomial $\chi$ the point 0 as root with multiplicity $k$, i.e. we take $\chi^*(z) = \chi(z)z^k$, then we obtain the same conclusion as in Proposition 3.4.
We are now able to give a semi-algebraic description of the set of real coefficients polynomials of a given degree which have one pair of roots, \(i\omega\) and \(-i\omega\), on the imaginary axis and no other root with zero real part.

**Theorem 3.5.** Let \(\chi(z) \in \mathbb{R}[z]\) be a degree \(n\) polynomial and write
\[
\chi(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n = \chi_1(z^2) + z \chi_2(z^2)
\]
with \(a_0 > 0\). Let \(\Delta_1, \Delta_2, \ldots, \Delta_n\) be the Hurwitz determinants sequence of \(\chi\). Then \(\chi(z)\) has a pair of distinct roots, \(i\omega\) and \(-i\omega\), on the imaginary axis and no other root with zero real part if and only if
\[
\Delta_{n-1} = 0, \quad a_n \Delta_{n-2} \Delta_{n-3} > 0.
\]

**Proof.**

\(\Rightarrow\) Let us suppose that the polynomial \(\chi(z)\) has a pair of roots, \(i\omega\) and \(-i\omega\), on the imaginary axis and no other root with zero real part. One then has \(\Delta_{n-1} = 0\) and \(\Delta_{n-3} \neq 0\) according to Corollary 3.2. On the other hand, let
\[
\chi(z) = (a_0 z^{n-2} + a_1 z^{n-3} + \cdots + a_{n-2}) (z^2 + \omega^2) = \chi^*(z^2 + \omega^2).
\]
One then has the relations \(a_n = a_{n-2} \omega^2\) and \(a_{n-2}^* = \frac{\Delta_{n-2}}{\Delta_{n-3}}\), which give the relation \(\omega^2 = a_n \frac{\Delta_{n-3}}{\Delta_{n-2}}\). Since \(\omega^2\) is positive, one obtains \(a_n \frac{\Delta_{n-3}}{\Delta_{n-2}} > 0\), which is equivalent to \(a_n \Delta_{n-2} \Delta_{n-3} > 0\).

\(\Leftarrow\) Let us suppose that \(\Delta_{n-1} = 0\) and \(a_n \Delta_{n-2} \Delta_{n-3} > 0\). Then according to Corollary 3.2 the polynomial \(\chi(z)\) has a pair of symmetric roots \(z_1\) and \(-z_1\). Moreover, one has the relation \(z_1^2 = -a_n \frac{\Delta_{n-3}}{\Delta_{n-2}}\), which proves that \(z_1^2 < 0\), and hence the symmetric roots of the polynomial \(\chi\) are on the imaginary axis.

As consequence of Theorem 3.1 and Proposition 3.4 we also obtain a nice semi-algebraic description of the set of real coefficients polynomials of a given degree which have all their roots in the left half-plane except one pair, \(i\omega\) and \(-i\omega\), on the imaginary axis.

**Theorem 3.6.** Let \(\chi(z) \in \mathbb{R}[z]\) be a degree \(n\) polynomial and write
\[
\chi(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n = \chi_1(z^2) + z \chi_2(z^2)
\]
with \(a_0 > 0\). Let \(\Delta_1, \Delta_2, \ldots, \Delta_n\) be the Hurwitz determinants sequence of \(\chi\). Then \(\chi(z)\) has a pair of distinct roots, \(i\omega\) and \(-i\omega\), on the imaginary axis and all the other roots in the left half-plane if and only if
\[
a_n > 0, \Delta_{n-1} = 0, \quad \Delta_{n-2} > 0, \ldots, \Delta_1 > 0.
\]

**Proof.**

\(\Rightarrow\) Let us suppose that the polynomial \(\chi(z)\) has a pair of roots, \(i\omega\) and \(-i\omega\), on the imaginary axis and the remainder of the roots in the left half-plane. One then has \(\Delta_{n-1} = 0\) according to Corollary 3.2. On the other hand, let
\[
\chi(z) = (a_0 z^{n-2} + a_1 z^{n-3} + \cdots + a_{n-2}) (z^2 + \omega^2) = \chi^*(z^2 + \omega^2).
\]
Since all the roots of the polynomial $\chi^*$ are in the left half-plane and following the Routh–Hurwitz criterion we obtain the signs conjunction

$$\Delta_{n-2}^* > 0, \ldots, \Delta_1^* > 0,$$

where the $\Delta_i^*$’s are the Routh–Hurwitz determinants of the polynomial $\chi^*$. According to Proposition 3.4 the $\Delta_i^*$’s are nothing but the $\Delta_i$’s, and thus we have the signs conjunction

$$\Delta_{n-1} = 0, \quad \Delta_{n-2} > 0, \ldots, \Delta_1 > 0.$$ 

Moreover, one has $a_n = a_{n-2}^*\omega^2$ and $a_{n-2}^* > 0$. It then follows that $a_n > 0$.

"⇐" Let us now suppose that $a_n > 0$, $\Delta_{n-1} = 0$, $\Delta_{n-2} > 0, \ldots, \Delta_1 > 0$.

Following Corollary 3.2 the polynomial $\chi$ factors in the form $\chi = \chi^*(z^2 - z_1^2)$, where $\chi^*$ is a real coefficients polynomial having no symmetric roots and $z_1^2$ is a real number.

According to the Routh–Hurwitz criterion and Proposition 3.4, the polynomial $\chi^*$ has all its roots in the left half-plane.

The fact that the pair of symmetric roots of the polynomial $\chi$ is purely imaginary is an immediate consequence of Theorem 3.5. 

### 3.3. Semi-algebraic description of Hopf bifurcation fixed points

Let us return to our parameterized vector field $f(u, x)$ and the autonomous ordinary differential system associated with it. Using Theorem 3.6, we are now able to give a semi-algebraic description of the set of parameters values for which a Hopf bifurcation (with empty unstable manifold) occurs for the system. Indeed, this can be expressed by the following first-order formula:

$$\exists x(f_1(u, x) = 0, f_2(u, x) = 0, \ldots, f_n(u, x) = 0, a_n > 0, \Delta_{n-1} = 0, \Delta_{n-2} > 0, \ldots, \Delta_1 > 0).$$

In this formula, $a_n$ is $(-1)^n$ times the Jacobian determinant of the matrix $Df(u, x)$, and the $\Delta_i$’s are the $i$th Hurwitz determinants of the characteristic polynomial of the same matrix $Df(u, x)$. We can also give a semi-algebraic description of the set of parameters values for which the system undergoes at most a Hopf bifurcation and all the rest of its eigenvalues are in the left half-plane. In terms of logical formulas this can be expressed as

$$\forall x((f_1(u, x) = 0, \ldots, f_n(u, x) = 0) \Rightarrow (a_n > 0, \Delta_{n-1} \geq 0, \Delta_{n-2} > 0, \ldots, \Delta_1 > 0)).$$

**Nondegenerate Hopf bifurcation fixed points**

The last point we discuss in this section is how to decide whether a given Hopf bifurcation fixed point $(u, x)$ is not degenerate with respect to a given parameter $u_i$ in the parameters list $u$. Here the nondegeneration means that the real part of the two conjugated eigenvalues of the matrix $Df(u, x)$ which cross the imaginary axis at $u_i$ has a nonzero partial derivative with respect to the parameter $u_i$. This condition ensures, according to the Poincaré–Andronov–Hopf theorem, the existence of small amplitude periodic solutions of the system when the parameter $u_i$ is subjected to a small perturbation near $u_i$ and the rest of the parameters remain unchanged.
PROPOSITION 3.7. Let \( f(u, x) \) be a parameterized polynomial vector field that undergoes a Hopf bifurcation at a fixed point \((u_0, x_0)\) with \( \omega \) and \(-\omega\) as imaginary eigenvalues. In a small connected neighborhood of the point \( u \) let \( \phi \) be the unique differentiable function of \( u \) such that \( f(u, \phi(u)) = 0 \) and \( \phi(u) = x \). Also, let \( \lambda(u) \) be the unique differentiable function of \( u \) such that \( \lambda(u) \) is a root of the characteristic polynomial \( \chi(u, \phi(u))(z) \) of the Jacobian matrix \( Df(u, \phi(u)) \) and \( \lambda(u) = i\omega \). If we let
\[
\chi(u, x)(z) = (z^2 + \omega^2)\chi^*(u, x)(z),
\]
then
\[
\frac{\partial \Delta_{n-1}}{\partial u_i}(u, x) = 2\Delta_{n-3}(u, x)\text{Res}(\chi^*(u, x), z^2 + \omega^2)\frac{\partial \text{Re}(\lambda)}{\partial u_i}(u, x).
\]

PROOF. Let us factor the polynomial \( \chi(u, \phi(u))(z) \) into the form
\[
\chi(u, \phi(u))(z) = (z^2 - 2\text{Re}(\lambda(u))z + \lambda(u)\bar{\lambda}(u))\chi^*(u, \phi(u))(z).
\]
According to Orlando's formula, cf. Gantmacher (1959), one has
\[
\Delta_{n-1} = 2\text{Re}(\lambda(u))\Delta_{n-3}^* \text{Res}(\chi^*(u, \phi(u)), \quad z^2 + 2\text{Re}(\lambda(u))z + \lambda(u)\bar{\lambda}(u)),
\]
where \( \Delta_{n-3}^* \) is the Hurwitz determinant of order \( n - 3 \) of the polynomial \( \chi^*(u, \phi(u)) \).

Since the coefficients of the polynomial \( \chi^*(u, \phi(u)) \) are differentiable in terms of \( u \) and since \( \text{Re}(\lambda(u)) = 0 \), one obtains
\[
\frac{\partial \Delta_{n-1}}{\partial u_i}(u, x) = 2\Delta_{n-3}^* \text{Res}(\chi^*(u, x), z^2 + \omega^2)\frac{\partial \text{Re}(\lambda)}{\partial u_i}(u, x).
\]
Finally, we have \( \Delta_{n-3}^*(u, x) = \Delta_{n-3}(u, x) \) according to Proposition 3.4, and so this gives the desired formula. \( \square \)

3.4. SIMPLIFICATION BY GRÖBNER BASIS METHODS

In many physical problems the vector field \( f = (f_1, \ldots, f_n) \) generates an ideal \( I = I(f_1, \ldots, f_n) \) such that the \( \mathbb{K} \)-algebra \( \mathbb{K}[u, x]/I \) is free of finite rank as \( \mathbb{K}[u] \)-module. This is the case, for example, when an \( n \)th order scalar autonomous equation
\[
x^{(n)} = P(x, x', \ldots, x^{(n-1)}),
\]
where \( P \) is a monic polynomial with respect to \( x^{(n-1)} \), is transformed into the ordinary autonomous system
\[
\begin{align*}
x' &= x_1 \\
x'_1 &= x_2 \\
&\vdots \\
x'_{n-1} &= P(x, x_1, \ldots, x_{n-1}).
\end{align*}
\]
This additional condition allows us to reduce the first-order formulas occurring in our study of Hopf bifurcation by eliminating all but one of the quantifiers, using a parameterized rational univariate representation for \( \mathbb{K}[u, x]/I \), see Gonzalez-Vega et al. (1999). This can be done by Gröbner basis techniques, which are usually faster than the general quantifier elimination methods. Once this is done, one obtains a first-order formula with only one quantifier.
However, if the system has more than two parameters, the formula obtained becomes too large to be successfully treated by the current quantifier elimination algorithms. Moreover, the formula obtained may contain rational functions in terms of the parameters, and thus must be transformed before the quantifier elimination step. To describe precisely how the system is transformed, let \( v \) be the parameter of the rational univariate representation and let \( Q(u, v) \neq 0 \) be an atomic formula, where \( \epsilon \in \{=, <, >\} \), occurring in our first-order formula. If \( q(u) \) is the common denominator of the coefficients of \( Q(u, v) \) when viewed as a polynomial in the variable \( v \), then the sign condition \( Q(u, v) \epsilon 0 \) must be replaced by a disjunction of signs conditions taking into account the sign of the polynomial \( q(u) \).

4. A Software-component Architecture

In this section we will describe the software-system architecture that we use to implement the algorithms.

The reduction of the questions on the differential equations to quantifier elimination problems are implemented in Maple. As a general-purpose algebra system, Maple and its symbolic library provide a convenient environment for these reductions.

Many of the necessary functions—such as converting systems of ODEs or computing their Jacobians are already included in the Maple library. Other functions, such as computing the sub-resultant sequences, can be implemented in the Maple language relatively close to their mathematical content.

However, an algorithm for quantifier elimination on real closed fields is not available in Maple. Implementing one in Maple ourselves or waiting for somebody else to implement it or only using a system in which one is implemented is only a sub-optimal solution, because of the following reasons.

(1) The task of quantifier elimination is very coarse grained, so the cost of a client–server communication to a remote quantifier elimination system are negligible in general.

(2) The existing quantifier elimination systems such as QEPCAD (Hong, 1990; Brown, 1998) or REDLOG (Dolzmann and Sturm, 1999) are tuned up considerably. Thus, implementing the general algorithms as described in the literature\(^1\)—which is still a considerable implementation effort—will give much worse results than using the existing tuned implementations.

As an example, the reader might want to consider the quantifier elimination functionality recently implemented in Mathematica 4. This functionality, which is mainly based on the cylindrical algebraic decomposition algorithm, cannot handle partially quantified formulas and is thus quite rudimentary compared to QEPCAD or REDLOG.

Hence our idea has been the following: we provide a system infrastructure, which allows us to use the “best available” software systems for quantifier elimination as if these algorithms were directly written as Maple libraries.

The system infrastructure should give the following abstractions.

\(^1\)See e.g. Collins (1975), Hong (1990), Collins and Hong (1991), and Brown (1998) for descriptions of the methods used in QEPCAD and Weispfenning (1994, 1997) for the ones used in REDLOG.
A Java-based architecture, which gives this flexibility, was introduced in Weber et al. (1998). There the general architecture was introduced for the example of a parallel Gröbner basis system. Various refinements of this general architecture have been achieved in the meanwhile. Some of them are described in Göbel et al. (1999). The refined architecture used for the computations given in Section 5 will be sketched below.

4.1. OVERVIEW OF THE JAVA-BASED ARCHITECTURE

An overview of the client side and server side of the architecture is given in Figure 1. On the client side there are alternative possibilities to access the server software: either from a specialized Java graphical user interface (GUI) (usually in the form of an applet) or out of a general-purpose algebra system, which calls the server side via the so-called “Java client adapter”. Up to now we have not implemented a special Java GUI for the quantifier elimination server, but this possibility has been used to access our parallel Gröbner basis software, see Weber et al. (1998).

The Java client adapter is a small Java program which has to be installed on the client
machine. This Java program has to be called from the algebra system (via a system command) and will communicate with the algebra system via pipes or files depending on the capabilities of the algebra system. We have included all these communication possibilities in our Java program.

The data in this communication are in the format of the algebra system and will be transformed by the Java client adapter into an internal exchange format. We currently use the exchange format of the MathBus (Zippel, 1997), but the MathBus could be substituted by another exchange format such as OpenMath (Dahmas et al., 1997; PolyMath Development Group, 1997) without too much effort. We have currently implemented parsers for the MathBus and MathML (World Wide Web Consortium, 1997), Maple, Mathematica and REDLOG. The conversion to QEPCAD is currently achieved via the QEPCAD interface implemented in REDLOG.

We have implemented the Java client adapter as a so-called Java Bean (Hamilton, 1997), which allows us to have a user configurable persistent state of the software. We mainly use this feature to allow a user to choose another server for the quantifier elimination software instead of our default server that we provide at the University of Tübingen.

For the communication between the client side and the call of the server side functions we use the standard Java RMI mechanism (Sun Microsystems, 1997). The Java server side adapter accepts the RMI request of the client. In general, it will transform the data out of the MathBus format to the format used by the server system and then use these data in a call of the server system. It will handle the result given by the server system and check whether an exception should be thrown or whether the result can be transformed into MathBus format, in which case it will be returned to the server.

The Java server adapter can call the server system by any means that are offered by Java: through the Java native interface or by calling processes via the Runtime classes, i.e. the Java process interface.

Thus the called system can be a parallel or distributed system itself. Moreover, the client adapter can call and coordinate different server systems itself thus giving new possibilities for distributed computing on the server side.

The specifics of this general scheme for the quantifier elimination server will be sketched below.

4.2. THE SERVER SIDE

A schematic view of the server side for the quantifier elimination system is given in Figure 2.

The Java server adapter will call the quantifier elimination systems in their native environments. It can be configured to choose the host with the least load out of a pool of server computers and to use competitive parallelism between REDLOG and QEPCAD. Using the multi-threading capabilities of Java this feature could be implemented quite easily.

For many examples, REDLOG finds an equivalent quantifier-free formula much faster than QEPCAD. However, REDLOG cannot handle all examples, there is a degree restriction on the quantified polynomials. If this degree restriction cannot be overcome (e.g. by polynomial factorization) REDLOG will return the formula it currently encountered, which is an equivalent partially quantified formula in general. Such an output from REDLOG can be given as an input to QEPCAD, which in general is easier than the original one.

In addition, in the cases in which REDLOG succeeds, a postprocessing of the result
by QEPCAD is very often advantageous. The equivalent quantifier-free formulas are very often very large—we have examples for which the answer of REDLOG was bigger than one megabyte in its printed form! However, QEPCAD can be used to simplify quantifier-free formulas—especially through its extensions described in Brown (1998). For many examples QEPCAD could simplify the answer of REDLOG quite considerably. Thus, using QEPCAD for postprocessing the answers given by REDLOG is currently our default setting. In this default setting, on the server side we use competitive parallelism of this combined system with QEPCAD alone, cf. Figure 2. Since these two programs run on different cpus, in no case is there a loss of performance using competitive parallelism rather than only one of them; so we left the competitive parallelism as a default setting although in almost all the examples we computed, the combination of REDLOG and QEPCAD has been the winner in the competition with QEPCAD alone.

Thus, using the idea of software components which can be plugged together (and also on the server side) gave us much better results than we could have obtained by using only one of the packages. Moreover, we can easily plug in improved or specialized quantifier elimination components to be developed in the future with minor changes on the server side and without changing the software on the client side at all.

Moreover, any new algorithmic improvements in the field of quantifier elimination on real closed fields can be readily tested by all examples generated on the client side. Thus we hope that our research might stimulate further work on quantifier elimination.

5. Computational Examples

The software which was used to compute the following examples is available at http://www-sr.informatik.uni-tuebingen.de/Projekt_WiSoft.html. There is the possi-
bility of installing the client side code only. The client side consists of the Maple library, which also contains examples, together with the Java client adapter. In this case the client will connect to the quantifier elimination server that is provided in Tübingen. We also provide the Java code for the server side, which enables a user to set up a server accessing QePCad and REDLOG on a site of the user’s choice.

5.1. Canonical example for Hopf bifurcation

**Example 5.1.** Let us consider the planar system

\[
\begin{align*}
\dot{x}(t) &= (du + a(x(t)^2 + y(t)^2))x(t) - (w + cu + b(x(t)^2 + y(t)^2))y(t) \\
\dot{y}(t) &= (w + cu + b(x(t)^2 + y(t)^2))x(t) + (du + ax(t)^2 + y(t)^2)y(t).
\end{align*}
\]

This system can be viewed as a typical system undergoing a Hopf bifurcation at the origin (0, 0). Indeed, if a given n-dimensional system \( \dot{x} = f(u, x) \), with a real parameter \( u \) undergoes a Hopf bifurcation at the origin when \( u = 0 \), then using normal forms techniques after projection on the center manifold (see Chow and Hale, 1996, for normal form techniques), one is reduced to studying a system of the form above with \( a, b, c, d, w \) given specified values.

A computation using our implementation gives the following signs conditions for the system to undergo a Hopf bifurcation with empty unstable manifold.

\[
0 < d^2u^2 + w^2 + c^2u^2 + 2wcu \land -2du = 0.
\]

Let us note that if \( d \neq 0 \), then the above formula is equivalent to

\[ u = 0. \]

This computation can be done via the Gröbner basis simplification alone, so that no call of the quantifier elimination server is necessary.

5.2. A chemical reaction system

**Example 5.2.** The following example comes from chemical reactions, see Chow and Hale (1996, p. 360)

\[
\begin{align*}
\dot{x}(t) &= a - (b + 1)x(t) + x(t)^2y(t) \\
\dot{y}(t) &= bx(t) - x(t)^2y(t).
\end{align*}
\]

In addition, in this example the computations can be performed on the client alone and give the condition

\[ a^2 - b + 1 = 0. \]

5.3. Use of Gröbner basis simplification

In fact, these two examples are immediate because of the use of a Gröbner basis simplification that can be used in our implementation. In these two examples, an easy computation gives \([x, y]\) as a Gröbner basis (for the lexicographical ordering) of the ideal generated by the components of the vector field. Thus, this allows us to avoid the quantifier elimination step by replacing \( x \) and \( y \) by 0 in the first-order formula.
partial quantifier elimination by Gröbner basis simplification

In other cases, Gröbner basis simplification allows us to eliminate only some of the quantifiers. In this case, usual quantifier elimination algorithms are used to eliminate the remaining quantifiers. In the following example, one quantifier is eliminated by the Gröbner basis simplification and the other is eliminated in the quantifier elimination step.

Example 5.3. Consider the following system, which is also one of the standard examples for Hopf bifurcations, cf. Chow and Hale (1996).

\[
\begin{align*}
\dot{x}(t) &= x(t)(1 + 1/4a^2 - 1/4(x(t) - 1 - a)^2 - y(t)) \\
\dot{y}(t) &= y(t)(x(t) - 1).
\end{align*}
\]

For this system, our implementation takes some seconds to give the signs condition \(a = 0\).

5.4. More complicated examples

A system arising in epidemiology

The following example is from Liu and van den Driessche (1995). In this research paper the investigation on the existence of Hopf bifurcations is an important part. The differential equations come from epidemiological models with varying population size and dose-dependent latency period.

Example 5.4. The following parameterized system of differential equations describes the so-called SEIS† models of Liu and van den Driessche (1995)

\[
\begin{align*}
\dot{s}(t) &= b - b s(t) + \gamma i(t) - (\beta - \alpha)s(t)i(t) \\
\dot{e}(t) &= -b e(t) + \beta s(t)i(t) + \alpha i(t)e(t) - \epsilon e(t) \\
\dot{i}(t) &= -(b + \gamma + \alpha)i(t) + \alpha i(t)^2 + \epsilon e(t).
\end{align*}
\]

In Liu and van den Driessche (1995) it is proved that this system does not have a Hopf bifurcation for any parameter values for the epidemiological relevant cases: all parameters and variables are positive and \(s(t) + e(t) + i(t) = 1\).

Using our software, the quantifier elimination programs did not succeed for the general system with three variables and five parameters within 1 day of computation time. The Gröbner basis simplification implemented in Maple did not succeed because of an “object too large” error. Without using this simplification (by setting the corresponding flag in the Maple program) our program could generate the first-order formula, which consisted of 14 atomic formulas involving three existentially quantified variables. REDLOG could eliminate two of the three quantified variables within a few seconds of cpu time, but failed to eliminate the third quantifier because of its degree restriction. This result given by REDLOG to QEPCAD could not be solved by QEPCAD within 1 day of cpu time. Also, the original problem could not be solved by QEPCAD in the same time. Thus both competing branches on the server side failed with this time restriction.

†SEIS stands for susceptibles (S), which can become exposed (E), i.e. are infected but not yet infectious, which will become infectious (I), which then become susceptibles (S) again.
When specializing four of the five parameters with various values, the combination of Redlog and Qepcad returned the correct result, namely false, within a few seconds of computation time.

LORENZ SYSTEM

Example 5.5. The so-called “Lorenz System” (Lorenz, 1963; Rand and Armbruster, 1987; Guckenheimer and Holmes, 1990) is given by the following system of ODEs:

\[
\begin{align*}
\dot{x}(t) &= \alpha (y(t) - x(t)) \\
\dot{y}(t) &= r x(t) - y(t) - x(t) z(t) \\
\dot{z}(t) &= x(t) y(t) - \beta z(t).
\end{align*}
\]

It is named after Edward Lorenz at MIT, who first investigated this system as a simple model arising in connection with fluid convection.

Applying our program to the Lorenz system imposing positivity conditions on the parameters gives the following answer after a few seconds of computation time:

\[
\alpha^2 + \alpha \beta - \alpha \tau + 3\alpha + \beta \tau + \tau = 0 \land \tau = 0 \land \alpha - \beta^2 - \beta \geq 0 \land 2\alpha - 1 \geq 0 \land \beta > 0.
\]

Thus we have found a simple closed from description involving three free parameters, which coincides (after some elementary transformation) with the result of a manual computation given in Guckenheimer and Holmes (1990).

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