Quantifier Elimination on Real Closed Fields and Differential Equations

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abstract. This paper surveys some recent applications of quantifier elimination on real closed fields in the context of differential equations. Although polynomial vector fields give rise to solutions involving the exponential and other transcendental functions in general, many questions can be settled within the real closed field without referring to the real exponential field.

The technique of quantifier elimination on real closed fields is not only of theoretical interest, but due to recent advances on the algorithmic side including algorithms for the simplification of quantifier-free formulae the method has gained practical applications, e.g., in the context of computing threshold-conditions in epidemic modeling.

1 Introduction
Differential equations are ubiquitous in real world problems modeling. Often one has to determine their trajectories from their initial conditions. Even in the simplest case of one-dimensional linear problems

\[ \frac{d}{dt} x(t) = ax(t) \]  
\[ x(t_0) = x_0 \]

the solutions involve the exponential function:

\[ x(t) = x_0 \cdot e^{a(t-t_0)}. \]

So the study of the field of the real numbers with the exponential function as a primitive is important for investigating differential equations. Nevertheless, we will not focus on the remarkable results obtained for the real exponential field, see, e.g., [M03] for a survey also discussing some of these

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results. Instead, we will show that many questions in the area of differential equations can be settled within the real closed field without referring to the real exponential field.

This possibility is due to the fact that many questions on dynamical systems can be posed by referring to the vector field only, e.g., the equilibrium points can be defined using the vector field only. Consider, e.g., the autonomous vector valued system

\[
\frac{d}{dt} x(t) = f(x(t)),
\]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( x : \mathbb{R} \to \mathbb{R}^n \). The set of equilibrium points of this system is the set of the zeros of the vector field \( f \), i.e., \( \{ e \in \mathbb{R}^n \mid f(e) = 0 \} \). Very often the vector field \( f \) is in the form of a parameterized polynomial vector field, \( f(x) = f(u, x) = (f_1, \ldots, f_n) \), where \( f_i \in \mathbb{R}[u, x] \) are polynomials of degree \( \leq d \), \( x = (x_1, \ldots, x_n) \) is a list of variables and \( u = (u_1, \ldots, u_k) \) is a list of parameters. Then the set of equilibrium points is algebraic over the parameters \( u \). The question whether there are equilibrium points can be formulated as an existentially quantified first-order formula in the language of ordered fields and in the theory of the ordered field of the reals. A quantifier-free equivalent formula can be found algorithmically, a famous result due to Tarski [T51].

Although the theoretical significance of these results is widely seen, very often there are doubts about their practical feasibility [S03]:

So, quantifier elimination is something that is do-able in principle, but not by any computer that you and I are ever likely to see. Well, I'll retract that last statement because it's probably false.

In this paper we do not only want to survey results showing that the technique of quantifier elimination on real closed fields is of theoretical interest for differential equations, but that due to recent advances on the algorithmic side including algorithms for the simplification of quantifier-free formulae the methods have gained practical applications, e.g., in the context of computing threshold-conditions in epidemic modeling.

2 Quantifier Elimination for Real Closed Fields

2.1 A brief history

Tarski's work on a decision method for elementary algebra and geometry [T51] is important for model theory in many aspects. In the survey paper of Macintyre [M03] it is contrasted to "Tarski's set-theoretic foundational formulations" being the starting point of

a quite different development, which still flourishes and owes very little to the set-theoretic development.
However, from a purely algorithmic point of view Tarski’s method is rather prohibitive, as its complexity cannot be bound by a tower of exponential functions, i.e., is not even elementary recursive. This asymptotic complexity is also the one of the methods described by Seidenberg [S54] and Cohen [C69]. The first elementary recursive method was found by Collins [C75] using the technique of Cylindrical Algebraic Decomposition (CAD), whose complexity is doubly exponential, thus reducing the complexity from an unbounded tower of exponentials to one of height two.

This is a provable lower bound for the general problem of quantifier elimination on real closed field [D88, W88]. More precisely, the lower double exponential bound is on the number of changes of quantifiers. For purely existentially or universally quantified problems methods of single exponential complexity have been described, e.g., by Renegar [R92].

The results on the worst-case asymptotic complexity do only give partial information about the running times for many concrete instances. A major breakthrough for practically working quantifier-elimination methods have been the so called “virtual substitution” methods. Based on ideas of Ferrante and Rackoff for decision problems [FR79], virtual substitution methods for quantifier elimination date back to a theoretical paper by Weispfenning [W88]. They have been devised for problems involving polynomials that are linear (or at most quadratic or cubic) in the quantified variables [W97, W94, L93]. Implementations of these methods are available in the REDLOG system mainly developed by A. Dolzmann, A. Seidl, and T. Sturm.1

Weispfenning [W99] also showed that the elementary theory of the real numbers in the language having 0, 1 as constants, addition and subtraction and integer part as operations, and equality, order and congruences modulo natural number constants as relations admits an effective quantifier elimination procedure and is decidable. He also showed that this so called “mixed real-integer linear quantifier elimination” sample answers for existentially quantified variables. Moreover, it comprises as special cases linear elimination for the reals, and Presburger arithmetic of asymptotically optimal complexity.

There are also sophisticated implementations of the cylindrical algebraic decomposition available in REDLOG and in the QEPCAD library,2 which contain substantial improvements in many aspects [A84, C91, B98, S03, D04].

Another technique for quantifier elimination on real closed fields was published by Weispfenning in 1998 [W98] (whereas a technical report describ-
ing the method had already been published by him in 1993 and the method was implemented as a Diploma thesis by A. Dolzmann in 1994). It is based on real root counting and is now called Hermitian quantifier elimination to acknowledge Hermite’s work in the area of real root counting.

Although the worst-case asymptotic complexity of this method is not elementary recursive such as Tarski’s method, it has been proved to be a powerful tool for particular classes of elimination problems, e.g., problems involving one quantifier block in front of a conjunction containing as many equations as quantifiers and only few other atomic formulae.

None of these methods is superior to another one in general, and some problems could be solved only by their combination, e.g., Dolzmann [D299] found an automatic solution of a real algebraic implicitization problem of the so called “Enneper surface” by combining all of these three quantifier elimination methods, namely quantifier elimination by virtual substitution, Hermitian quantifier elimination, and quantifier elimination by partial cylindrical algebraic decomposition, as well as the simplification methods described in [D2S597].

2.2 Simplifications of Quantifier-Free Formulae

The same semi-algebraic set can be represented by different quantifier-free formulae. On the equivalence classes of quantifier-free formulae describing the same semi-algebraic sets there are different reasonable (partial) orderings giving the notion of one formula being “simpler” than another one, e.g., is $a < 0 \land b = 0 \lor a < 0 \land c = 0$ simpler than $a < 0 \land [b = 0 \lor c = 0]?$

However, many formulae produced by automated systems like the quantifier elimination package REDLOG, or by substitution and specialization of rules like the Routh-Hurwitz criterion, are large and complex, even when the objects they define are quite simple. It is important to consider simplification of formulae, if these have to be used in further computations or made available for human comprehension. Fortunately, the simplification on this level turns out to be on a much coarser level then the one considered above.

There are two algorithmic techniques that we are aware of for simplification of large quantifier-free formulae—which both define in their way a notion of one formula being “simpler” than another one. One technique is described in [D2S597] and implemented in REDLOG, and the other is described in [B198] and implemented in the SLFQ system.\footnote{Available at http://www.cs.usna.edu/~qepcad/SLFQ/Home.html.} The goals of the two methods are very different. The former is intended primarily to combat intermediate expression swell during virtual term substitution. As such it needs to be fast. The latter is intended to reduce the size of the formula, in
terms of number of irreducible polynomials appearing, as much as possible, regardless of time required to do so.

The SLFQ system uses the QEPCAD as a black box to do formula simplification. QEPCAD is able to simplify formulae, but its time and space requirements become prohibitive when input formulae are large. SLFQ basically breaks large input formulae into small pieces, uses QEPCAD to simplify the pieces, and starts a process of combining simplified subformulae and applying QEPCAD to simplify the combined subformulae. Eventually this process produces a simplification of the entire initial formula. QEPCAD takes a formula and constructs an explicit geometric model of the object that formula defines in real Euclidean space—a Cylindrical Algebraic Decomposition (CAD). Using the CAD, it is easy to detect when one of those varieties does not actually define a boundary of the geometric object. Once detected it can be easily removed from the CAD, resulting in a simpler CAD representing the same object. This can be repeated until we reach a minimal CAD—i.e., a CAD from which no polynomial can be removed without violating the requirement that the CAD represents the same geometric object. This CAD simplification process is described in [B98].

The user may also allow SLFQ to produce a simplified formula that disagrees with the input formula, but only on a set of points that is measure zero in the space of all variable assignments—a similar idea as the one behind the so called generic quantifier elimination [D2SW98, SS93, DG04]. By allowing SLFQ this limited degree of error, the time and space requirements of its computations can be dramatically reduced, and in some cases simpler formulae may be found. For situations in which variables have physical interpretations, allowing this limited error makes sense, since no physical parameter can be controlled precisely enough to be constrained to a measure zero set.

The following examples, which are taken from [B1ENW04], demonstrate how some of these switches affect SLFQ’s results. The following calls are with the same input formula.

```
% cat infile
[r - t > 0 /\ r + t > 0] / [r - t < 0 /\ r + t > 0]
% ./slfq infile -q
r - t /= 0 /\ r + t > 0
% ./slfq infile -q -a "r>0 /\ t>0"
r - t /= 0
% ./slfq infile -q -a "r>0 /\ t>0" -F
TRUE
```

The first call just simplifies the given formula ("-q" tells SLFQ to run
in a “quiet” mode). In the second call both variables are assumed to be positive. In the third call both variables are assumed to be positive and SLFQ is allowed to produce a simplified formula that disagrees with the input, but only on a measure zero subset of the space of all variable assignments.

Refer to §3.2 to see SLFQ applied to large, complex input formulae that arise from applying the Routh-Hurwitz criterion to parameterized equilibrium points, and which can be reduced by SLFQ to small and meaningful formulae.

3 Investigating Equilibrium Points

Formally in the following we will only deal with time independent dynamical systems given by polynomial vector fields. This class covers a wide range of practical applications. Moreover, we can in a certain sense define some functions arising as solutions of differential equations by polynomial vector fields of a higher-dimension.

For a given equilibrium point \( \mathbf{x} \) of a \( C^\infty \) vector field \( f \) the study of the system near this point is classically done by Taylor expanding \( f \) near \( \mathbf{x} \) and considering at first the linear system

\[
\frac{d}{dt} \zeta = D(f)(\mathbf{x}) \cdot \zeta, \quad (5)
\]

where \( D(f)(\mathbf{x}) \) is the Jacobian matrix of \( f \) at the point \( \mathbf{x} \). When the matrix \( D(f)(\mathbf{x}) \) is hyperbolic, i.e., it has no eigenvalue with zero real part, then the stability study of the nonlinear system near the point \( \mathbf{x} \) reduces to the study of the stability of the linear system near the origin \( 0 \). In the presence of eigenvalues with zero real part, the linear system gives only partial information about the local dynamics of the nonlinear system near the point \( \mathbf{x} \). In fact, the local behavior near \( \mathbf{x} \) of the nonlinear system depends on the higher order terms of the Taylor expansion of \( f \) near the point \( \mathbf{x} \). However, the number of eigenvalues with zero real part of \( D(f)(\mathbf{x}) \) remains a fundamental invariant in the study of the topological nature of the local dynamics near the point \( \mathbf{x} \). A systematic way to deal with non-hyperbolic situations is to use center manifold techniques and normal forms theory (see e.g., [G3H390, C2H396]).

3.1 Existence of equilibria points

Given the parametric nature of the system of differential equations even the question of the existence of equilibria points is a non-trivial question. For many applications this question not only reduces to the one of multi-dimensional equation solving, but to the one of solving the equations with inequality conditions.
Although this is a conceptually simple idea the following example might show some of the power of currently available systems for quantifier-elimination on real closed fields.

**An example.**
As an example we will take a system arising in epidemic modeling. In this context inequality constraints arise naturally. Consider the SEIRS model [L1v95], which has also been investigated by quantifier-elimination methods in [B1E2N2W106].

The SEIRS-model for the transmission of infectious diseases is given by the following system of 4 ordinary differential equations:

\[
\begin{align*}
\frac{d}{dt}S &= \mu + \gamma R - \mu S - \beta IS \\
\frac{d}{dt}E &= \beta IS - (\mu + \sigma)E \\
\frac{d}{dt}I &= \sigma E - (\nu + \mu)I \\
\frac{d}{dt}R &= \nu I - (\mu + \gamma)R
\end{align*}
\]  

(6) (7) (8) (9)

The informal meaning of the variables and parameters is as follows:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>susceptibles</td>
</tr>
<tr>
<td>E</td>
<td>exposed (not yet infectious)</td>
</tr>
<tr>
<td>I</td>
<td>infectious</td>
</tr>
<tr>
<td>R</td>
<td>recovered (currently immune)</td>
</tr>
<tr>
<td>(\beta)</td>
<td>transmission parameter</td>
</tr>
<tr>
<td>(\mu)</td>
<td>birth rate = mortality rate</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>rate of change from exposed to infectious</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>rate of loss of immunity</td>
</tr>
<tr>
<td>(\nu)</td>
<td>rate of loss of infectiousness</td>
</tr>
</tbody>
</table>

A point in SEIR-space is an equilibrium point if

\[
0 = \mu + \gamma R - \mu S - \beta IS \land 0 = \beta IS - (\mu + \sigma)E \land 0 = \sigma E - (\nu + \mu)I \land 0 = \nu I - (\mu + \gamma)R
\]  

(10)

and represents an endemic state if

\[
S > 0 \land E > 0 \land I > 0 \land R > 0.
\]  

(11)

Therefore, there is an endemic equilibrium for the SEIRS-model if there exist real numbers \(S, E, I, R\) such that both formulae (10) and (11) hold.
Note that, as there are several different specialized and general quantifier elimination methods available, it is by no means clear cut as to how this is best done. In situations in which it applies, the method of virtual term substitution is generally much faster than CAD-based quantifier elimination. On the other hand, its output formulae are often extremely large. For this example virtual term substitution does apply so the approach discussed in [B1E2N2W106] is to perform quantifier elimination by virtual term substitution and simplify the result using SLFQ.

The input to REDLOG is the following:

\[ A := \mu + \gamma R - \mu S - \beta J S = 0 \]
\[ \text{and } \beta J S - (\mu + \sigma) F = 0 \]
\[ \text{and } \sigma F - (\nu + \mu) J = 0 \]
\[ \text{and } \nu J - (\mu + \gamma) R = 0; \]

\[ F := \text{ex}(\{S,F,J,R\}, A \text{ and } F > 0 \text{ and } J > 0 \text{ and } R > 0 \text{ and } S > 0); \]

\[ C := \{\beta > 0, \nu > 0, \sigma > 0, \gamma > 0, \mu > 0\}; \]

\[ G := \text{rlqe}(F,C); \]

Note that variables \( E \) and \( I \), which have special meaning in Reduce, have been replaced with \( F \) and \( J \) respectively. REDLOG computes a quantifier-free equivalent formula \( G \) consisting of 25 atomic formulae. Given the assumptions on the parameters, SLFQ simplifies this to

\[ \sigma \beta - \sigma \nu - \mu \nu - \mu \sigma - \mu^2 > 0 \]

with the following input:

\[ \text{slfq} -a "\beta > 0 /\ \nu > 0 /\ \sigma > 0 /\ \gamma > 0 /\ \mu > 0" \ G \]

Notice that this formula does not contain all parameters; there is no dependency on \( \gamma \), the rate of loss of immunity. Using the assumption on the parameter—using that \( \sigma \) is positive—this condition is equivalent to

\[ \beta > \frac{(\nu + \mu)(\sigma + \mu)}{\sigma} \]

the threshold condition obtained by “hand computations” in the epidemiological literature.
3.2 Testing stability for equilibrium points

Let \( f(u, x) = (f_1, \ldots, f_n) \) be a parameterized vector field, where \( f_i \in \mathbb{R}[u, x] \) are polynomials of degree \( \leq d \), \( x = (x_1, \ldots, x_n) \) is a list of variables and \( u = (u_1, \ldots, u_k) \) is a list of parameters. Let us consider the autonomous ordinary differential system

\[
\frac{d}{dt} x = f(u, x)
\]  

(14)

and let us denote by \( \Phi_t(u, x) \) the flow generated by the vector field \( f \). A good place to start the study of the nonlinear system \( \frac{d}{dt} x = f(u, x) \) is to find its equilibrium points, which are given by the equation

\[
f(u, x) = 0.
\]

(15)

If the list of parameters \( u \) is given a value \( u_0 \in \mathbb{R}^k \), and \( (u_0, x) \) is an equilibrium point of the specialized nonlinear system \( \frac{d}{dt} x = f(u_0, x) \), the study of the behavior of the flow \( \Phi_t(u_0, x) \) when starting near the equilibrium point \( (u_0, x) \) is classically done using the linear system

\[
\frac{d}{dt} \zeta = D(f)(u_0, x) \cdot \zeta
\]

(16)

where \( D(f)(u_0, x) \) is the Jacobian matrix of the vector field \( f(y, x) \) at the point \( x \). The flow generated by this linear system is then \( e^{tD(f)(u_0, x)} \cdot \zeta \) = \( D(\Phi_t)(u_0, x) \cdot \zeta \).

A fundamental result due to Hartman and Grobmann (see, e.g., [A173]) states that in the case of hyperbolic equilibrium points, i.e., the matrix \( D(f)(u_0, x) \) has no eigenvalue with zero real part, the nonlinear flow has the same behavior near the equilibrium point \( (u_0, x) \) as the linear flow near the origin 0. In particular, the nonlinear flow \( \Phi_t(u, x) \) is asymptotically stable near the equilibrium point \( (u_0, x) \) if and only if all the eigenvalues of the matrix \( D(f)(u_0, x) \) have negative real part.

According to the well known Routh-Hurwitz criterion, see e.g. [H4L97] this last condition is equivalent to the signs conjunction

\[
\Delta_1(u_0, x) > 0 \land \cdots \land \Delta_n(u_0, x) > 0,
\]

(17)

where the \( \Delta_i(u_0, x) \)'s are the Hurwitz determinants associated to the characteristic polynomial of the matrix \( D(f)(u_0, x) \).

As the nonlinear system \( \frac{d}{dt} x = f(u, x) \) is parameterized, a natural question is to ask for which values \( u \) of the parameter \( u \) the specialized system \( \frac{d}{dt} x = f(u, x) \) is asymptotically stable near all its equilibrium points. This can be symbolically expressed by the following first-order formula:

\[
\forall x \ (f(u, x) = 0 \Rightarrow \Delta_1(u, x) > 0 \land \cdots \land \Delta_n(u, x) > 0).
\]

(18)
One can also ask for which values $u$ of the parameter $u$ the specialized system $\dot{x} = f(u, x)$ is asymptotically stable near at least one of its equilibrium points:

$$\exists x \ (f(u, x) = 0 \land \Delta_1(u, x) > 0 \land \cdots \land \Delta_n(u, x) > 0). \quad (19)$$

These questions, as many others, are thus reduced to quantifier elimination problems for first-order formulae in the language of real closed fields.

**Stability of specific parameterized equilibria points.**

In some applications there exist specific equilibrium points for all parameter values of interest. However, the stability of these equilibrium points depends on the parameters. Although these problem is not a quantifier elimination problem per se, as applying the Routh-Hurwitz criterion to the Jacobian at the specific equilibrium point already does contain the parameters only and thus there are no quantified variables.

Nevertheless, the formula obtained by the Routh-Hurwitz criterion is huge and beyond human comprehension in general, whereas in many cases arising from applications it describes a rather simple object. The CAD-based simplification techniques realized in the slfq program (see §2.2) have proven to be a very useful tool also for this purpose. This statement shall be exemplified by the following example, which is taken from [B1E2N2W106], where also some subtleties related to the possibility of being a non-hyperbolic equilibrium point are discussed.

Consider the SEIRS-model from above. For all parameter values it has the equilibrium point $S = 1, E = 0, I = 0, R = 0$, which can be interpreted as the “disease free equilibrium”. Applying the Routh-Hurwitz criterion to the Jacobian at this point gives a quantifier-free formula containing a large polynomial of degree 9 consisting of 203 terms.

Using the positivity condition on all parameters slfq can simplify this large formula to the following one:

$$\sigma \beta - \sigma \nu - \mu \nu - \mu \sigma - \mu^2 < 0. \quad (20)$$

The required computation time has been 0.15 seconds in slfq. Notice that this formula coincides with the negation of the formula asking for “endemic equilibrium” for the SEIRS model (12) modulo the measure zero set involving equality. The fact that the condition for the local stability of the disease-free equilibrium and the existence of an endemic equilibrium partition the space of valid parameters tells us that there are no parameter values that produce a so called “sub-threshold endemic equilibrium” for these models cf. [B1E2N2W106, H097, vW02, K0V00].
Remark on “hand computations” done in the epidemiological literature. In the epidemiological literature the stability of the “disease free equilibrium” is very often the starting point for computing threshold conditions. However, very often the threshold conditions are formulated using the concept of basic reproduction ratio $R_0$, which denotes the number of secondary infections from each infected individual. This concept, first introduced by Dietz [D75], is also applicable for stochastic models. If $R_0$ exceeds one the disease it will reach an endemic stage, in which the disease is always present in the population, if it less than one it will die out.

In general experts in the field have calculated threshold conditions on the basis of $R_0$ “by hand”, either solely by paper and pencil, or in part using computer algebra systems such as Maple or Mathematica as “symbolic calculators”. However, in [C1M3W94] the QEPcad system for quantifier elimination on real closed field has been used to parametrically investigate $R_0$ for a model of the epidemic of the AIDS disease.

Remark on vaccination policies. One of the parameters that can be influenced by change of behavior is the transmission parameter $\beta$ (or its variants). By estimating the transmission parameter from empirical data and using the estimates for the other parameters out of the medical literature one can see how far away from the threshold one is. A symbolic computation of the threshold condition has the major advantage that the influences of changes on the parameters can be estimated much better than would be the case by numerical estimates. This might be one of the reasons why previously a lot of work involving tedious “hand calculations” have be spent to obtain symbolic threshold conditions.

Another possibility to come below the threshold is to reduce the number of susceptibles by vaccinations. If a proportion $p$ of newborns is vaccinated then it can be easily shown by a simple change of variables for most of the models we are considering—such as the SEIRS model—that the effect on the dynamics is the same as if in the original model the transmission parameter $\beta$ is replaced by $\beta(1-p)$. We refer to [E0R5B0G200] for the details in the case of the SEIR model. Thus by vaccinating a sufficiently high fraction of susceptibles it is possible to avoid infections also in the group of remaining susceptibles. In the case of RSV epidemics [W1W2M01, N2W105] the numerical value of the threshold for $\beta$ is about 35, when using the disease specific values for average latency period, average duration of infectiousness and the birth-rates for developed countries. The estimates of $\beta$ for pre-vaccination epidemics are ranging from 70 to 240 for different locations (and variations of the model). Thus the critical percentages of vaccinations are ranging from 50% to about 85% for this example.
3.3 Testing Stability by Quantifier Elimination

A wide variety of stability questions for differential equations—ordinary differential equations, ordinary discrete difference equations, initial-boundary value problems for partial differential equations, and semi-discrete equations—is reduced to first-order formulae in the language of the ordered-field of the reals in $[H_4L_0S_397]$. Also the local stability of Runge-Kutta discretizations is investigated.

In $[H_4L_0S_397]$ the quantifier elimination is performed by Qepcad. Simple problems could be solved in a few seconds, and most textbook examples in some minutes or at least a few hours of computation time. However, they found relatively modest problems that were beyond the reach of direct solutions by Qepcad.

3.4 Bifurcations

As parameters are varied in a given parameterized dynamical system, the phase portrait may undergo qualitative changes. The parameter values where such changes occur and the corresponding changes are called bifurcations. One of the main goals of bifurcations theory is the location of those parameters regions in which a given dynamical system displays the desired behavior.

The simplest bifurcations take place at equilibrium points, and they are called local bifurcations. For a given equilibrium $(u, x)$ a bifurcation may arise when the matrix $D(f)(u, x)$ has some eigenvalues with zero real part. In this case, and for $(u, x)$ close enough to $(u, x)$ radically new dynamical behavior can occur. For example, equilibrium points can be created or destroyed, and even new orbits such as periodic or quasi-periodic ones can be created.

In general, for an $n$-dimensional autonomous system there are many distinct bifurcating situations depending on the number of eigenvalues with zero real part. A partial classification of local bifurcations is done by using the concept of codimension. For example, codimension one bifurcations are of two kinds: either the Jacobian matrix has a zero eigenvalue or a pair of pure imaginary eigenvalues. In the first case we have a Saddle-node bifurcation and the second case corresponds to the so-called Hopf bifurcation.

At a Saddle-node bifurcation a pair of equilibrium points coalesce one another. On one side of the bifurcation in the parameter space there are two equilibrium points, and they disappear on the other side. When the system undergoes a Hopf bifurcation at a equilibrium point $(u, x)$, and the parameters $u$ are subjected to small perturbations, the original equilibrium point $(u, x)$ moves analytically in terms of $u$ and no new equilibrium is created in the neighborhood. However, if the imaginary eigenvalues of the
linearized system move away from the imaginary axis, one expects the equilibrium point to change its stability type. This change is typically marked by the appearance of a small periodic orbit encircling the equilibrium point as stated by the Poincaré-Andronov-Hopf theorem, see e.g. \cite{C2H196}. The local dynamics near an equilibrium point with Hopf bifurcation cannot be determined by the linear approximation of the vector field. In fact, depending on the nonlinear terms of $f$, the equilibrium point can be unstable, stable or even asymptotically stable.

Semi-algebraic characterizations of Hopf bifurcations.

El Kahoui and Weber \cite{E2W100} showed that Hopf bifurcation fixed points have a semi-algebraic description. The description is carried out by use of the Hurwitz determinants. Applying techniques from the theory of subresultant sequences and of Gröbner bases they could to come up with efficient reductions, which lead to quantifier elimination questions that can often be handled by existing quantifier elimination packages.

The result of the reduction is as follows: For a parameterized vector field $f(u, x)$ and the autonomous ordinary differential system associated with the semi-algebraic description of the set of parameters values for which a Hopf bifurcation (with empty unstable manifold) occurs for the system can be expressed by the following first-order formula:

$$\exists x (f_1(u, x) = 0 \land f_2(u, x) = 0 \land \cdots \land f_n(u, x) = 0$$

$$\land a_n > 0 \land \Delta_{n-1} = 0 \land \Delta_{n-2} > 0 \land \cdots \land \Delta_1 > 0).$$

In this formula $a_n$ is $(-1)^n$ times the Jacobian determinant of the matrix $Df(u, x)$, and the $\Delta_i$'s are the $i$th Hurwitz determinants of the characteristic polynomial of the same matrix $Df(u, x)$.

It is also possible to give a semi-algebraic description of the set of parameters values for which the system undergoes at most a Hopf bifurcation and all the rest of its eigenvalues are in the left half-plane. In terms of logical formulae this can be expressed as follows:

$$\forall x ((f_1(u, x) = 0 \land \cdots \land f_n(u, x) = 0) \Rightarrow$$

$$\quad (a_n > 0 \land \Delta_{n-1} \geq 0 \land \Delta_{n-2} > 0 \land \cdots \land \Delta_1 > 0)).$$

Example of computations for Hopf bifurcation fixed points.

The following examples are taken from \cite{E2W100}. The quantified formula expressing the condition for the Hopf bifurcation fixed point is computed in Maple. Using a software component architecture—which was also used to connect other mathematical services \cite{W1K1E198, G1K1M4W199}—the quantifier elimination was then performed by a combination of REDLOG
and QEPCAD for simplifying the results of REDLOG. In [E2W100] and in more detail in [E2W102] it is also shown how to simplify the (partially) quantified formulae by Gröbner basis techniques.

**Canonical example for Hopf bifurcation.** The following planar system can be viewed as the typical system undergoing a Hopf bifurcation at the origin \((0,0)\):

\[
\begin{align*}
\frac{dx}{dt} & = (du + a(x(t))^2 + y(t)^2))x(t) - (w + cu + b(x(t))^2 + y(t)^2))y(t), \\
\frac{dy}{dt} & = (w + cu + b(x(t))^2 + y(t)^2))x(t) + (du + ax(t)^2 + y(t)^2))y(t).
\end{align*}
\]

(23)

(24)

If a given \(n\)-dimensional system \(\frac{d}{dt} x = f(u,x)\), with a real parameter \(u\) undergoes a Hopf bifurcation at the origin when \(u = 0\), then using normal forms techniques after projection on the center manifold (see [C2H196] for normal form techniques), one reduces to study a system of the form above with \(a, b, c, d, w\) given specified values.

The computation reported in [E2W100] gives the following signs conditions for the system to undergo a Hopf bifurcation with empty unstable manifold:

\[
0 < d^2 u^2 + w^2 + c^2 u^2 + 2 w c u \land -2 d u = 0.
\]

(25)

Notice that in the case \(d \neq 0\) the above formula is equivalent to the following simple formula:

\[
u = 0.
\]

(26)

**A system arising in epidemiology.** The following example is from [L1v95]. In this research paper the investigation on the existence of Hopf bifurcations is an important part. The differential equations come from epidemiological models with varying population size and dose-dependent latency period.

The following parameterized system of differential equations describes the so called SEIS models of [L1v95]:

\[
\begin{align*}
\frac{ds}{dt} & = b - b s(t) + \gamma i(t) - (\beta - \alpha) s(t) i(t), \\
\frac{de}{dt} & = -b e(t) + \beta s(t) i(t) + \alpha i(t) e(t) - \varepsilon e(t), \\
\frac{di}{dt} & = -(b + \gamma + \alpha) i(t) + \alpha i(t)^2 + \varepsilon e(t).
\end{align*}
\]

(27)

(28)

(29)

In [L1v95] it is proved that this system does not have a Hopf bifurcation for any parameter values for the epidemiological relevant cases: all parameters and variables are positive and \(s(t) + e(t) + i(t) = 1\).
In \([E_2W_100]\) it is reported that the quantifier elimination programs did not succeed for the general system with 3 variables and 5 parameters within one day of computation time. When specializing 4 of the 5 parameters with various values, the combination of REDLOG and QEPCAD returned the correct result, namely \texttt{false}, within some seconds of computation time.

Using the refined implementations of the methods in the current version of REDLOG a recently performed quantifier elimination on the formula was successful within some seconds of computation time. However, the computation resulted in a large quantifier free formula in the 5 parameters (consisting of 236 atomic subformulae). This formula should be equivalent to \texttt{false}, i.e., no fulfilling instances of the variables should exist. Unfortunately, SLFQ was not able to do the simplification of this formula. Because of the high degree of the polynomials quantifier elimination on the existential quantification of this formula can not use the currently available virtual substitution methods. A CAD based quantifier elimination on the existential quantification of this formula had to go through the entire tree to find that there are no fulfilling instances. This task was also attempted but not successfully finished by REDLOG within two days of computation time and 512 MB of main memory.

**Lorenz system.** The famous “Lorenz System” \([L_263, G_3H_390, R_1A_087]\) is given by the following system of ODEs:

\[
\begin{align*}
\frac{dx}{dt} &= \alpha (y(t) - x(t)) \quad (30) \\
\frac{dy}{dt} &= r x(t) - y(t) - x(t) z(t) \quad (31) \\
\frac{dz}{dt} &= x(t) y(t) - \beta z(t) \quad (32)
\end{align*}
\]

It is named after Edward Lorenz at MIT, who first investigated this system as a simple model arising in connection with fluid convection.

After imposing positivity conditions on the parameters, the following answer is reported in \([E_2W_100]\) (requiring some seconds of computation time then):

\[
\begin{align*}
\alpha^2 + \alpha \beta - \alpha r + 3 \alpha + \beta r + r &= 0 \land \\
\alpha r - \alpha - \beta^2 - \beta &\geq 0 \land \\
2\alpha - 1 &\geq 0 \land \beta > 0. \quad (33)
\end{align*}
\]

Thus a simple closed form description involving three free parameters has been found algorithmically by the use of quantifier elimination on real closed fields, which coincides (after some elementary transformation) with the result of a hand computation given in \([G_3H_390]\).
4 Testing for Ellipticity of Partial Differential Equations

An important task in the theory of partial differential equations [R2R493] is their classification into elliptic and hyperbolic systems. The distinction of these two classes is fundamental not only for the theory but also for the numerical analysis, as it decides what kind of conditions (initial or boundary) should be imposed. Furthermore, their solutions behave very differently. Notice that we do not treat parabolic systems separately here. At the coarse level of the discussion here, parabolicity is a degenerate case of hyperbolicity and only appears when finer notions like strict hyperbolicity are introduced.

For elliptic systems boundary value problems are usually well-posed and their solutions show typically a very high regularity. From an application point of view, they model stationary problems. In hyperbolic systems a distinguished direction (“time”) exists and one considers initial value problems for them; thus they represent models for evolutionary problems. Even for regular data their solutions may exhibit shocks.

There are several different notions of ellipticity: classical ellipticity, Petrovsky ellipticity and the notion of ellipticity introduced by Dougis and Nirenberg [D3N55], which will be called DN-ellipticity in the following. DN-ellipticity is the most general of these.

Ellipticity of a system in general—and DN-ellipticity in particular—is defined at a point in the space of independent variables. Thus a given system may be elliptic at some points and not at other. Therefore, the answer to the question whether a system is elliptic will be a description of the region in the space of independent variables in which the system is elliptic.

Seiler and Weber [S2W03] showed that when the coefficients on the given system of PDEs are algebraic or rational functions, the problem of determining DN-ellipticity at a point can be phrased as a decision problem in the first-order theory of the ordered field of the reals. Characterizing the regions in which the system is DN-elliptic is a problem of quantifier elimination on real closed fields. Notice that the quantifier elimination allows finitely many symbolic constants that can also appear in the input system of PDEs.

The definition of DN-ellipticity involves the introduction of a set of integer weights. In the reduction suggested in [S2W03] these weights become part of the quantified formulae that are produced from an input system of PDEs. So this formulations leads to a mixed real-integer quantifier elimination problem of a type that can be solved by the methods of Weispfenning [W399]. In [S2W03] it is noted that (because of the special form of the
problem) an approach in which integer weights are treated as real variables produces the same result as the mixed real-integer quantifier elimination.

In [B1dN05] algorithms for determining regions of DN-ellipticity based on solving real quantifier elimination without the “non-intrinsic” variables are announced. Given the dependence on the complexity of quantifier elimination on the number of variables and quantifier alternation, this is a substantial performance gain.

**Tricomi equation.** The following example is taken from [S2W03]. Consider the following variation of Laplace’s equation, the Tricomi equation:

\[
\frac{\partial^2 u}{\partial y^2} - y \frac{\partial^2 u}{\partial x^2} = 0
\]  

(34)

Obviously, it is only elliptic for \(y < 0\). The logical formula for its DN-ellipticity is obtained in MuPAD as follows:

```plaintext
LDF := Dom::LinearDifferentialFunction(Vars=[[x,y],[u]],
Rest=[Types="Indep"]):
tricomi := LDF(u([y,y])-y*u([x,x])): LDF::ellCond(tricomi)
```

The result is the following first-order formula (in REDLOG syntax):

```plaintext
ex(s1, 
ex(t1, 
  (s1 <= 0) and 
  (0 <= t1) and 
  all(xi1, 
  all(xi2, 
  all(w1, 
  not( 
    ((xi1 <> 0) or (xi2 <> 0)) and 
    ((w1 <> 0)) and 
    ((((s1 + t1 = 2)) impl (-w1*(xi1**2*y - xi2**2) = 0)))
  )
)))
))))
```
The generated formula has one free parameter (the variable \(y\), as we are dealing with a variable coefficient equation), an inner block of three universally quantified variables and an outer block of two existentially quantified variables and consists of 7 atomic subformulae. The corresponding quantifier free formula was found by REDLOG in less than 1 sec of computation time and is exactly the one we expect, namely \(y < 0\).

In [S2W03] successful computations on examples involving 7 existentially and 6 universally quantified variables consisting of 68 atomic subformulae have been reported.

References.


Quantifier Elimination on Real Closed Fields


