

AN ADAPTABLE SURFACE PARAMETERIZATION METHOD

P. Degener, J. Meseth and R. Klein
University of Bonn
Institute of Computer Science II
Römerstrasse 164
D-53117 Bonn, Germany

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ABSTRACT

Parameterizations of triangulated surfaces are used in an increasing number of mesh processing applications for various purposes. Although demands vary, they are often required to preserve the surface metric and thus minimize angle, area and length deformation. However, most of the existing techniques primarily target at angle preservation while disregarding global area deformation.

In this paper an energy functional is proposed, that quantifies angle and global area deformations simultaneously, while the relative importance between angle and area preservation can be controlled by the user through a parameter. We show how this parameter can be chosen to obtain parameterizations, that are optimized for an uniform sampling of the surface of a model. Maps obtained by minimizing this energy are well suited for applications that desire an uniform surface sampling, like re-meshing or mapping regularly patterned textures.

Besides being invariant under rotation and translation of the domain, the energy is designed to prevent face flips during minimization and does not require a fixed boundary in the parameter domain. Although the energy is non-linear, we show how it can be minimized efficiently using non-linear conjugate gradient methods in a hierarchical optimization framework and prove the convergence of the algorithm.

The ability to control the tradeoff between the degree of angle and global area preservation is demonstrated for several models of varying complexity.

Keywords: parameterization, uniform sampling, metric, re-meshing

1. INTRODUCTION

Parameterization denotes the task of finding a two dimensional map for a surface in a higher dimensional space. In computer graphics such maps have recently gained much interest, since they are used in many applications ranging from re-meshing, texture mapping and surface reconstruction to 3d painting systems, surface editing [2] and geometry images [8].

Most of these applications demand parameterizing maps (in the following also called parameterizations) that preserve the metric structure of the surface, i.e.

respect area and angles of shapes. Unfortunately, in general such an angle and area preserving parameterization does not exist, thus angle preservation has to be traded off against area preservation. Many existing methods focus on angle preservation only, which often leads to large global area distortion resulting in visually disturbing artifacts on textured surfaces even if angular distortions are small.

Simultaneously optimizing angle and global area deformation is often superior as shown in figure 1. Furthermore combined global area and angle optimization is

also important whenever an uniform surface sampling is desired, as in 3D painting systems or surface editing applications. However, by now only few approaches considering both angular and area distortion simultaneously exist, all of which exhibit certain drawbacks.

1.1 Previous Work

Over the last years a lot of research has been done in the area of surface parameterization. Besides methods that optimize the parameterization for a given surface signal like Balmelli et al.[1] and Sander et al.[21], most approaches aim at minimizing a metric distortion.

In the context of parameterization, harmonic maps [6, 5] were first used by Eck et al.[4]. To compute harmonic maps, Eck et al. derive appropriate weights for a system of edge springs which can be efficiently solved. However, the texture coordinates for boundary vertices must be fixed a priori and harmonic maps may contain face flips (adjacent faces in texture space with opposite orientation) which violate the bijectivity of a parameterization. Based on earlier work by Tutte[27], Floater[7] proposes a different set of weights for the edge spring model that guarantees bijectivity if the texture coordinates of the boundary are fixed to a convex polygon. Desbrun et al.[3] define a space of measures spanned by a discrete version of the Dirichlet energy [19], and a discrete authalic energy. While the authalic energy remedies local area deformations, it requires fixed boundaries and results cannot achieve the quality of methods targeted at global length preservation such as Sander et al.[22].

In Hormann and Greiner[12] mostly isometric parameterizations are introduced that minimize a non-linear energy. A variant of this energy is also used in this paper. Mostly isometric parameterizations do not require boundary texture coordinates to be fixed and avoid face flips. Furthermore, mostly isometric param-

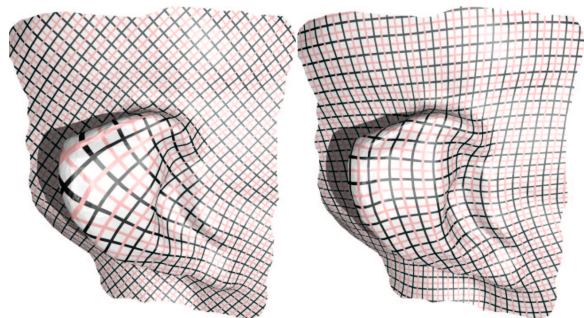


Figure 1: A regularly patterned texture mapped by a conformal map (left) and by a map computed with our method with $\theta = 1$ (right). While the conformal map minimizes angular distortion, the map on the right also takes global area deformation into account.

eterizations approximate mathematically well studied continuous conformal maps, i.e. maps that perfectly preserve angles.

Another approach to minimize angular distortion is proposed by Sheffer and de Sturler[24]. They define a non-linear energy in terms of the corner angles of the mesh in texture space. Lévy et al.[17] formulate the discrete conformality problem as an unconstrained quadratic minimization problem and prove the uniqueness and existence of its solution. Using a standard numerical conjugate gradient solver they are able to compute least squares approximations to continuous conformal maps very efficiently without requiring fixed boundary texture coordinates. However, in seldom cases triangle flips may occur.

In addition, some methods exist which compute parameterizations over a non planar domain. Haker et al.[9] compute conformal maps from a spherical domain onto a three dimensional surface. In Lee et al.[16] a mesh simplification [10] is used to parameterize a surface over a base mesh. A similar approach is taken by Khodakovsky et al.[14] but with emphasis on globally smooth derivatives.

Besides angle preserving methods, only a few approaches explicitly optimize global area or global length distortion: Maillot et al.[18] minimize an edge length distortion, but cannot guarantee the absence of face flips. The authors also propose an area preserving energy and combine both energies in a convex combination. Sander et al.[22] minimize the average or maximal singular value of the Jacobian to prevent undersampling of the surface. However, since they only penalize undersampling, oversampling of a triangle may nevertheless occur. To optimize for a uniform sampling Sorkine et al.[26] minimize the maximum of the maximal singular value and the inverse of the minimal singular value, which penalizes both under- and oversampling. While they obtain impressive results, their functional is not differentiable and thus not suitable for fast non-linear minimization techniques as the conjugate gradient method for example.

Iterative smoothing of an overlay grid is proposed by Sheffer and de Sturler[25] as a post-processing step for angle preserving parameterization algorithms. However, it is not clear what impact the post-processing has on the angle preservation.

1.2 Contribution

In this paper, we propose a metric energy that simultaneously measures angular and global area deformations imposed by a parameterization. On surfaces with non zero Gaussian curvature, the unavoidable deformation of angles and areas is traded off by the energy in an user-controlled way. Furthermore, we show how

this functional can be used to optimize parameterizations for a uniform surface sampling.

It is designed to prevent face flips during optimization and does not require fixed boundary texture coordinates. Furthermore it is invariant under rotation and translation of the domain. Although the derived energy is non-linear, it is differentiable and well suited for a hierarchical minimization as proposed by Hormann et al.[13]. We show how angle and global area optimized parameterizations can be computed efficiently with guaranteed convergence using non-linear conjugate gradient methods.

Usually models are cut into charts before being parameterized. In the present paper we do not tackle this problem, but our method can be combined with any charting and seaming algorithm available like the ones introduced in [23, 17, 22].

Besides face flips the bijectivity of the parameterization can also be violated if the texture mesh intersects itself. Although the method proposed here does not prevent these self intersections, they occur only in seldom cases and can be handled in a post processing step as proposed in [24].

2. ISOMETRIC DISTORTION

2.1 General Setup and Notation

Given an orientable 2-manifold surface patch $S \subset \mathbb{R}^k$ a parameterization is defined as a homeomorphism

$$\begin{aligned} \phi : \Omega \subset \mathbb{R}^2 &\rightarrow S \\ (u, v) &\mapsto \phi(u, v) \end{aligned}$$

from the parameter space Ω into S . In the following we consider the problem of finding a parameterization for a set S that has a triangulation

$$\mathcal{M}' = \{[1 \dots n], \mathcal{T}, (p_i)_{i=1 \dots n}\}$$

where $[1 \dots n]$ denotes the vertices, $\mathcal{T} \subset [1 \dots n]^3$ represents triangles and p_i is the location of vertex i in S . Furthermore, we require the inverse parameterization $\psi := \phi^{-1}$ to be linear within the triangles of $\mathcal{M}'$. Such a mapping ψ is uniquely determined by its values $((u_i, v_i))_{i=1 \dots n} := (\psi(p_i))_{i=1 \dots n}$ on the mesh vertices and

$$\mathcal{M} = \{[1 \dots n], \mathcal{T}, ((u_i, v_i))_{i=1 \dots n}\}$$

is a parameter domain triangulation for the image $\psi(S)$. The inverse parameterization ψ maps vertices and faces of $\mathcal{M}'$ onto vertices and faces of $\mathcal{M}$ respectively. In the following $\Delta_{\mathcal{M}'}((l, m, n))$ with $(l, m, n) \in \mathcal{T}$ denotes the triangle (p_l, p_m, p_n) in S . Analogously, $\Delta_{\mathcal{M}}(T)$ will be used to denote triangles in Ω .

Since a homeomorphism respects the topology and as we assume a planar domain $\Omega \subset \mathbb{R}^2$ the surface patch is required to have genus zero.

2.2 Measuring Distortion

Given a differentiable parameterization

$$\phi : \Omega \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^k$$

the first fundamental form $\mathbf{I}_\phi$, which captures the metric structure of S , is defined as

$$\mathbf{I}_\phi = \nabla^t \phi \cdot \nabla \phi = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $a = \left\| \frac{\partial \phi}{\partial u} \right\|^2$, $b = \left\langle \frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v} \right\rangle$ and $c = \left\| \frac{\partial \phi}{\partial v} \right\|^2$. Since $\mathbf{I}_\phi$ is a symmetric positive definite 2x2 matrix in every $\omega \in \Omega$ it induces a scalar product on $\mathbb{R}^2$ which describes the lengths and angles of vectors in $\mathbb{R}^2$ after being mapped by $\mathbf{I}_\phi$.

In the following we briefly review an angle preserving condition in terms of $\mathbf{I}_\phi$ and formulate a similar condition for global area preservation. In section 2.5 an energy functional $\mathbf{E}$ on the space of valid parameterizations is proposed that quantifies both angle and area deformation.

2.3 Conformal Maps

A result dating back to 1851 known as the Riemann mapping theorem guarantees for surface patches homeomorphic to a disk the existence of a conformal differentiable parameterization with continuous derivatives. A parameterization is said to be *conformal* if for every $\omega \in \Omega$

$$\mathbf{I}_\phi(\omega) = \lambda(\omega) \cdot \mathbf{I} \quad (1)$$

where $\mathbf{I}$ denotes the 2x2 identity matrix.

In other words the derivatives of the iso-u and iso-v curves passing through $\phi(\omega)$ are orthogonal and of the same magnitude. Thus conformal mappings preserve the angles. Denoting the maximal and minimal eigenvalue of $\mathbf{I}_\phi$ by λ_{max} and λ_{min} respectively, the conformality can equivalently be expressed as

$$\frac{\lambda_{max}}{\lambda_{min}} = 1$$

Since $0 < \lambda_{min} \leq \lambda_{max}$, one is the minimal value of the ratio of the eigenvalues and we choose to minimize this ratio to optimize angular distortion.

2.4 Area Distortion

The conformality condition allows the directional derivatives to be uniformly scaled by a factor $\lambda(\omega)$ that may vary if we travel from point to point on the surface. If this factor does not equal one, a shape in the

domain appears stretched or shrunk when mapped onto the surface and its area is distorted.

Since $\lambda(\omega)$ is continuous, around every $\omega \in \Omega$ a sufficiently small neighborhood exists, where the variation of $\lambda(\omega)$ is arbitrarily small. Thus, area is locally but not globally preserved by a conformal map. Conformal maps are therefore well suited for applications where angle preservation is required, but global area preservation is less important.

If in addition to angles, area is to be preserved globally, the magnitude of the directional derivatives has to be fixed leading to the notion of isometry. A parameterization is said to be *isometric* if

$$\lambda(\omega) = 1$$

for all $\omega \in \Omega$. Stated differently the first fundamental form equals the identity matrix in every point.

Isometry is stronger than conformality in the sense that it requires the tangent vectors to the iso parameter curves to be orthogonal and have unit length in every point of the surface. An isometric parameterization preserves angles and area globally. Unfortunately, isometric parameterizations exist only for surfaces with zero Gaussian curvature. In the general case of non zero Gaussian curvature, angle and area preservation have to be traded off.

To find the area deformation imposed by a map ϕ , we consider a sufficiently small axis aligned square in Ω of area A . The image of this square is a trapezoid spanned between the directional derivatives in u and v whose area is given by $A \cdot \sqrt{\det \mathbf{I}_\phi}$ and thus ϕ preserves area if and only if

$$\sqrt{\det \mathbf{I}_\phi} = 1$$

2.5 A Combined Energy

To enforce the area preservation condition proposed above, we choose $f(x) = x + \frac{1}{x}$ as objective function, since it is convex and attains its minimum in one. Furthermore it grows to infinity for both $x \rightarrow \infty$ or $x \rightarrow 0$. In the case of the area deformation energy

$$\mathbf{E}_{area}(\omega) := f(\sqrt{\det \mathbf{I}_\phi(\omega)}) = \sqrt{\det \mathbf{I}_\phi(\omega)} + \frac{1}{\sqrt{\det \mathbf{I}_\phi(\omega)}}$$

which is obtained by substituting $\sqrt{\det \mathbf{I}_\phi(\omega)}$ for x , this property ensures that the orientation of all faces is preserved during the minimization and thus face flips cannot occur.

Using the same objective function for the angle deformation yields the conformal energy

$$\mathbf{E}_{angle}(\omega) := f\left(\sqrt{\frac{\lambda_{max}}{\lambda_{min}}}\right) = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} + \sqrt{\frac{\lambda_{min}}{\lambda_{max}}}$$

that was proposed by Hormann in [11] and which is nothing but the MIPS energy that was used in [12] to compute angle preserving maps. The additional square root is used because the eigenvalues measure scale squared instead of scale.

Although a minimization of the area deformation energy alone is possible in theory, it causes severe numerical problems. The reason for this lies in the invariance of $\mathbf{E}_{area}$ under shears: Since a shear does not change the area of a triangle, during the optimization of $\mathbf{E}_{area}$ triangles may be arbitrarily sheared. Unfortunately such an extremely sheared triangle causes numerical problems in the minimization algorithm. We have thus decided to choose a combined energy as follows:

$$\mathbf{E}_{combined}(\omega) := \mathbf{E}_{angle}(\omega) \cdot (\mathbf{E}_{area}(\omega))^\theta$$

where the parameter θ varies between 0 and ∞ and controls the relative importance of area and angle preservation. Our algorithm was able to minimize the combined energy function at least for values of $\theta < 2$. However, for higher values of θ numerical problems prevented the minimization of the energy in some cases. These problems are due to very tall and narrow texture triangles caused by shearing. Minimizing the energy on such triangles has a bad condition.

For the special choice of $\theta = 1$, the combined energy becomes the simple product

$$\begin{aligned} \mathbf{E}_{angle}(\omega) \cdot \mathbf{E}_{area}(\omega) &= f\left(\sqrt{\frac{\lambda_{max}}{\lambda_{min}}}\right) \cdot f(\sqrt{\det \mathbf{I}_\phi(\omega)}) \\ &= f(\lambda_{max}) + f(\lambda_{min}) \end{aligned}$$

where the fact $\det \mathbf{I}_\phi = \lambda_{min} \cdot \lambda_{max}$ was used in the second equation. As the eigenvalues λ_{max} and λ_{min} measure the greatest and the smallest stretch respectively that the parameterization ϕ imposes on a vector of unit length, the energy obtained for $\theta = 1$ enforces an uniform sampling of the surface, and - similar to the energy proposed by Sorkine et al. in [26] - penalizes oversampling ($\lambda_{min} < 1$) as well as undersampling ($\lambda_{max} > 1$).

A parameterization ϕ can now be assigned a combined area and angle distortion by integrating over the surface patch S

$$\mathbf{E}(\phi) := \int_S \mathbf{E}_{combined}(\phi^{-1}(p)) dp$$

2.6 Discretization

For the special case of a piecewise linear parameterization over a triangulation, $\nabla \phi$ and $\mathbf{I}_\phi$ are constant within each triangle of $\mathcal{M}$ which in turn causes the energies $\mathbf{E}_{area}(\omega)$, $\mathbf{E}_{angle}(\omega)$ and $\mathbf{E}_{combined}(\omega)$ to be constant within each triangle.

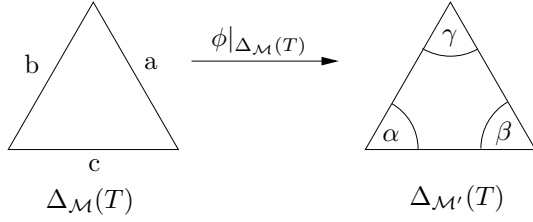


Figure 2: The restriction of ϕ to a triangle $\Delta_{\mathcal{M}}(T)$

As shown by Hormann in [11] the MIPS energy of the linear map $\phi|_{\Delta_{\mathcal{M}}(T)}$ can be written using the notation from figure 2 as

$$\mathbf{E}_{angle}(T) = \frac{\cot \alpha |a|^2 + \cot \beta |b|^2 + \cot \gamma |c|^2}{2 \text{area}(\Delta_{\mathcal{M}}(T))}$$

Furthermore we have for the linear map $\phi|_{\Delta_{\mathcal{M}}(T)}$

$$\sqrt{\det \mathbf{I}_{\phi|_{\Delta_{\mathcal{M}}(T)}}} = |\det \nabla \phi|_{\Delta_{\mathcal{M}}(T)}| = \frac{\text{area}(\Delta_{\mathcal{M}'}(T))}{\text{area}(\Delta_{\mathcal{M}}(T))}$$

and therefore the area distortion measure within a triangle is given by

$$\mathbf{E}_{area}(T) = \frac{\text{area}(\Delta_{\mathcal{M}'}(T))}{\text{area}(\Delta_{\mathcal{M}}(T))} + \frac{\text{area}(\Delta_{\mathcal{M}}(T))}{\text{area}(\Delta_{\mathcal{M}'}(T))}$$

Finally the integral becomes the finite sum

$$\mathbf{E}(\phi) = \sum_{T \in \mathcal{T}} \mathbf{E}_T \cdot \text{area}(\Delta_{\mathcal{M}'}(T)) \quad (2)$$

where

$$\mathbf{E}_T := \mathbf{E}_{angle}(T) \cdot \mathbf{E}_{area}(T)^\theta$$

only depends on the coordinates and texture coordinates of the three vertices in T .

2.7 Properties

The distortion measure $\mathbf{E}$ derived in the previous section has some important properties that should be briefly mentioned:

1. Invariance under Rotation and Translation

Since $\mathbf{E}$ is defined in terms of $\mathbf{I}_\phi$ which is in turn defined in terms of $\nabla \phi$, it is invariant under the translation $\phi(\omega + t)$ of the domain by a constant vector t . If the domain is transformed by some orthogonal transformation $\mathbf{R}$, the first fundamental form becomes

$$\mathbf{I}_{\phi \circ \mathbf{R}} = \mathbf{R}^t \nabla^t \phi \nabla \phi \mathbf{R},$$

i.e. its eigenvalues and determinant do not change. Depending only on the eigenvalues and the determinant of the first fundamental form, $\mathbf{E}$ is thus invariant under such a transformation $\mathbf{R}$. However, the distortion measure is not invariant under uniform scalings.

2. Differentiability The partial derivatives $\frac{\partial \mathbf{E}}{\partial u_i}$ and $\frac{\partial \mathbf{E}}{\partial v_i}$ exist for any valid piecewise linear parameterization, allowing for an efficient minimization of the functional in (u_i, v_i) .

3. Infinite Error for Degenerate Mappings For a valid parameterization, the mesh in the domain $\mathcal{M}$ contains no triangles degenerated to a point or a line and all faces are consistently oriented. If a triangle in $\mathcal{M}$ tends to degenerate, the parameterization has to stretch an infinitesimal small triangle onto the non-degenerated surface triangle. As λ_{max} measures the greatest stretch imposed on a vector of unit length it tends to infinity as a triangle tends to degenerate. This in turn causes both the MIPS energy and the area distortion energy to attain arbitrary high values.

Having this property, we can - following Sander et al.[22] - continue the error functor on degenerated configurations $((u_i, v_i))_{i=1 \dots n}$ by assigning them an infinitely high error. The minimization then automatically avoids such degenerated configurations, thus a consistent face orientation can be guaranteed. The property described above ensures that the continuation on degenerated configuration is continuous, which is essential for the numerical minimization.

3. MINIMIZING ISOMETRIC DISTORTION

3.1 Hierarchical Optimization

To minimize the non-linear isometric energy described in the previous section we use the hierarchical parameterization algorithm proposed by Hormann et al.[13]. A hierarchical approach is reasonable since it speeds up the computation and helps to circumvent local minima of the energy functional. Since in contrast to other energies boundary vertices do not need to be fixed, the proposed energy is well suited for a hierarchical optimization.

The method proposed by Hormann et al. computes a progressive mesh sequence [10] of $\mathcal{M}$, grouping independent splits in sets. These sets define a natural hierarchy for the optimization, with each set containing approximately 25% of the vertices of the subsequent stage. For further details on the generation of the split sets and the hierarchy please refer to Hormann et al.[13].

The actual optimization of the energy functional $\mathbf{E}$ on each level of the hierarchy uses a relaxation method, which is further described in the next section. Algorithm 1 shows a short overview over the basic steps of the algorithm.

Listing 1: Basic steps of the optimization algorithm

```
// build sets of independent splits
sets = buildSets();

// relax base mesh
mesh = baseMesh;
while(!convergence){
    relax(mesh);
}

for(int i=0;i<sets.size();i++){
    // apply splits of the next set to the mesh
    mesh.applyToMesh(sets[i]);

    // find save texcoords for new vertices
    generateSaveTexcoords();

    // relax mesh
    while(!convergence){
        relax(mesh);
    }
}
```

3.2 Vertex Relaxation

The texture coordinates (u_i, v_i) of vertex i affect only those $\mathbf{E}_T$ for which i is incident with T . More specifically only the partial sum

$$\mathbf{E}_i := \sum_{T \in 1\text{-ring}(i)} \mathbf{E}_T$$

of $\mathbf{E}$ is influenced by (u_i, v_i) .

Given an initial configuration $((u_i, v_i))_{i=1 \dots n}$, the vertex relaxation consists of two steps: First all vertices are ordered by the error $\mathbf{E}_i$ that they contribute to the overall error. Then for each vertex i $\mathbf{E}_i$ is optimized in (u_i, v_i) while keeping all other texture coordinates fixed.

Since the functional provides partial derivatives, the Polak Ribiere method [20] — a non-linear conjugate gradient optimizer — was used to optimize $\mathbf{E}_i$. This method includes a line search as a subtask, which was restricted to search for optimal vertex texture coordinates (u_i, v_i) only within the kernel of the vertex' 1-ring. The kernel of a polygon with a counterclockwise directed boundary is the intersection of all the half-planes lying to the left of the polygon's edges (see [16]). Since the faces of a mesh $\mathcal{M}$ in the plane are consistently oriented if and only if every vertex lies within the kernel of its 1-ring, the relaxation does not cause any face flips, provided that the initial configuration is free of flips.

Minimizing the partial sum $\mathbf{E}_i$ of $\mathbf{E}$ in each step, the relaxation decreases $\mathbf{E}$ monotonously and as the en-

ergy has a lower bound of zero, $\mathbf{E}$ is guaranteed to converge, ensuring that the 'while' loops in algorithm 1 terminate eventually.

3.3 Initial Vertex Placement

Since the surface patch is assumed to have genus zero, the simplification produces a base mesh that consists only of a single triangle. The texture coordinates of its vertices are initialized to a congruent triangle in the plane centered in the origin.

Whenever a set of splits is applied to the mesh during the hierarchical optimization, texture coordinates for the newly inserted vertices have to be found. In [13] barycentric coordinates obtained by exponential mapping are stored for each vertex during the simplification of the mesh. After a split these stored coordinates are used to assign a texture coordinate to the newly inserted vertex. However, in some cases the texture coordinate obtained this way is invalid, i.e. some of the triangles in the 1-ring are flipped. In order to ensure a valid configuration for the subsequent relaxation step, we take a different approach here.

To ensure a flipless configuration, the texture coordinate of the new vertex has to be inside the kernel of its 1-ring. One possible choice is certainly to use the center of the kernel as an initial texture coordinate for the new vertex. However, this choice requires the explicit computation of the 1-ring kernel. In contrast, the intersection of a polygon kernel with a line does not require the computation of the kernel. It is simply the intersection of the line with the halfplanes defined by the edges of the polygon.

To find a valid position for the new vertex, we repeatedly cast a ray from the texture coordinate of the vertex that is to be split in a random direction and intersect it with the 1-ring kernel. As soon as a non-empty intersection is found we choose the center of this intersection interval as an initial texture coordinate for the new vertex. Usually one or two kernel-ray intersections suffice to position the new vertex.

4. RESULTS

We applied our method to various models which are listed in table 1 together with the isometric distortion as measured by our energy. In addition, the L_2 and L_∞ distortions that were used in Sander et al.[22] are listed. The timings were taken on an AMD Duron 800MHz with 256MB memory.

Our initial motivation was to compute parameterizations that not only preserve angles but also minimize global area deformations. The results shown in Figure 4 were obtained for an angle/area weighting of $\theta = 1$. All of these surfaces have large areas of non zero

Model	Vertices	Error	L_2	L_∞	Stages	Time
MaxPlanck	25445	9.35	1.45	2.92	41	308.90
Venus	29322	4.55	1.21	1.77	41	300.46
Cat	4539	4.65	1.36	3.73	33	47.65
Horsehead	2893	4.85	1.36	3.53	32	33.79
Ear	2150	2.21	1.03	1.56	30	23.03

Table 1: Models and Statistics: In addition to our own energy, values of the L_2 and L_∞ energies are listed. Stages corresponds to the number of independent split sets. Errors were normalized to the surface area. For all models we chose $\theta = 1$.

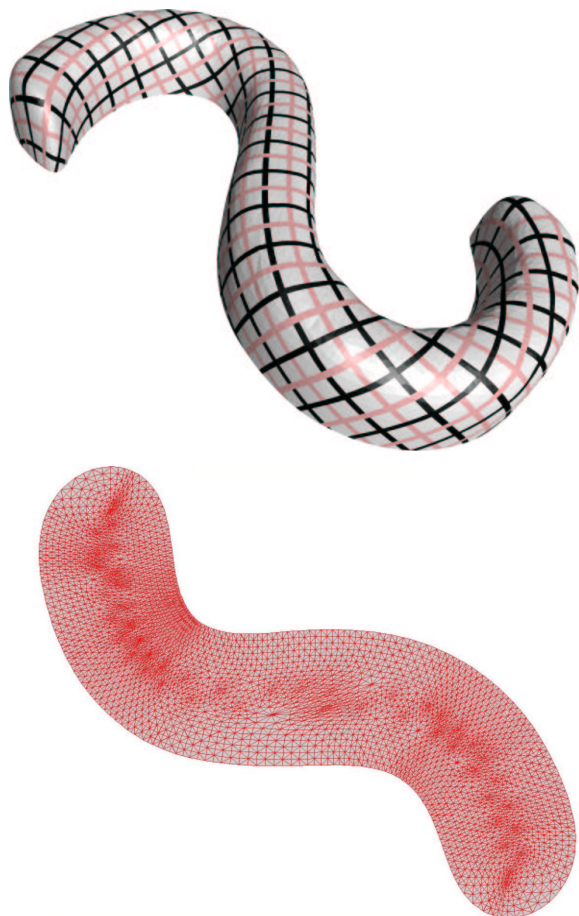


Figure 3: Parameterization obtained for a dented S-shaped model with non-convex boundary and the corresponding mesh in texture space

Gaussian curvature but only small boundary loops and thus can only be parameterized with high angle or area distortions. As a reference conformal maps were computed (right column of figure 4) using the LSCM method proposed in Lévy et al.[17]. The comparison shows that the maps obtained by our method trade part of the angle preservation to improve global area preservation as expected.

In Figure 5 the impact of the parameter θ on the parameterization is shown. As expected intuitively, a high value favors global area preservation, while smaller values emphasize the preservation of angles. For $\theta = 0$ the resulting map is similar to those obtained by the LSCM [17] or MIPS [12] method.

This observation can also be verified in the distortion histograms for the horse head dataset shown in figure 6. As in Lévy et al.[17] the area distortions were computed in each triangle as the ratio of texture area to model area. The angle histogram shows the distribution of the angles between the u and v directional derivatives in each triangle. For values of θ close to zero, the angle histogram shows a distinct peak, while the deviation in the area histogram is much higher. For higher values of θ this relation is reversed.

As mentioned above, the proposed energy does not require boundary vertices to be fixed. Thus, for any value of θ the minimization can also find an optimal boundary for $\mathcal{M}$. Figure 3 shows the parameterization obtained for a S-shaped model and the corresponding triangulation $\mathcal{M}$ in texture space.

In practice models are usually preprocessed by a chart or seam cutting algorithm before a parameterizing algorithm is applied. These cutting algorithms generate one or more charts with lower Gaussian curvature and larger boundaries which facilitate parameterization. But since the resulting parameterization heavily depends on the quality of the cutting method used, the parameterizations presented in this paper were obtained for uncut models. Only minor modifications were made to ensure a disk like topology. However, the preprocessing was only skipped for means of demonstration and the results certainly improve a lot by using a charting or seaming algorithm like those proposed in Sheffer and Hart[23] or Lévy et al.[17].

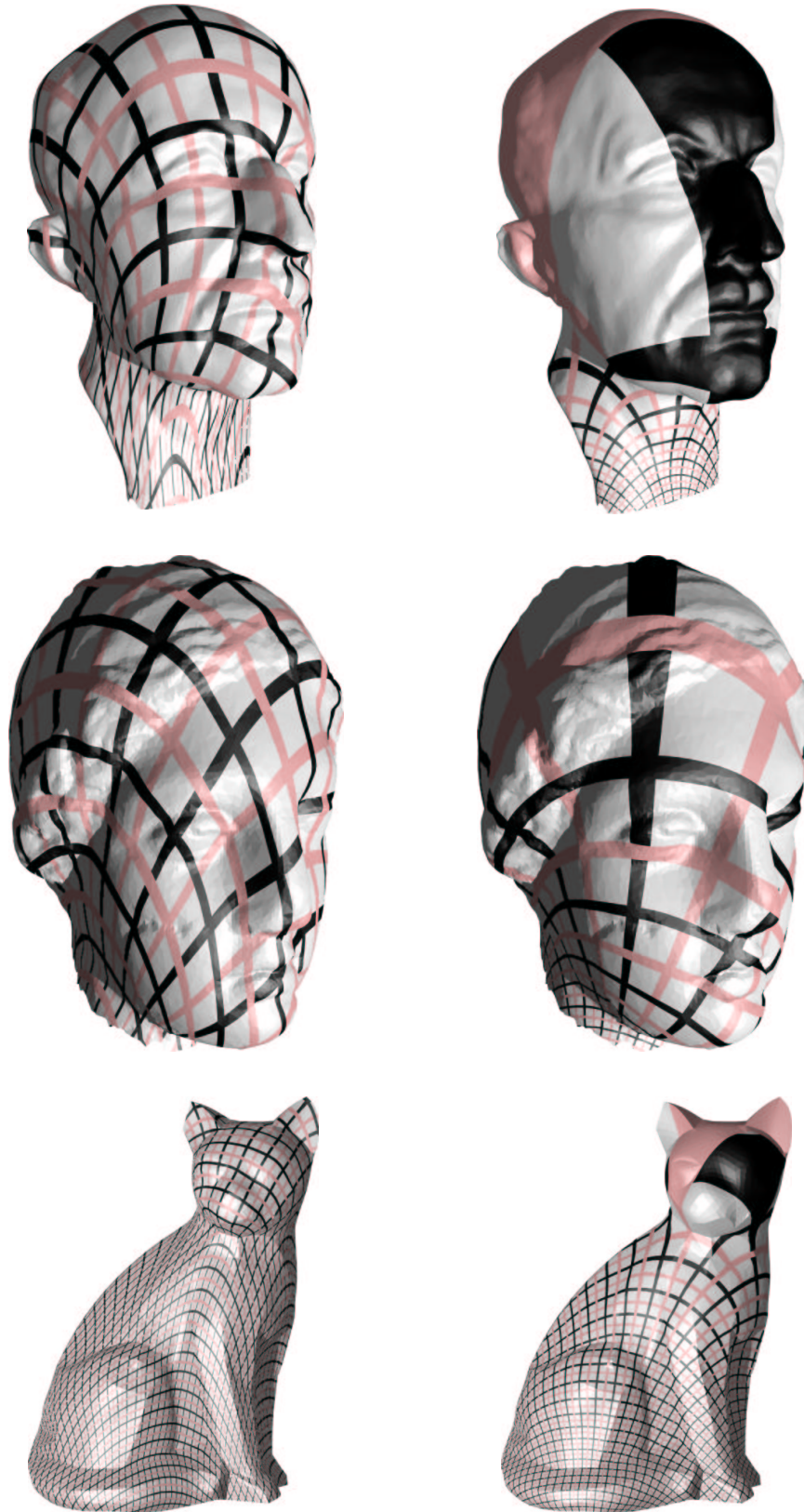


Figure 4: These models are parameterized as they appear, without charting or seaming. The column on the left shows the results obtained with our method for $\theta = 1.0$. On the right a conformal mapping is shown.

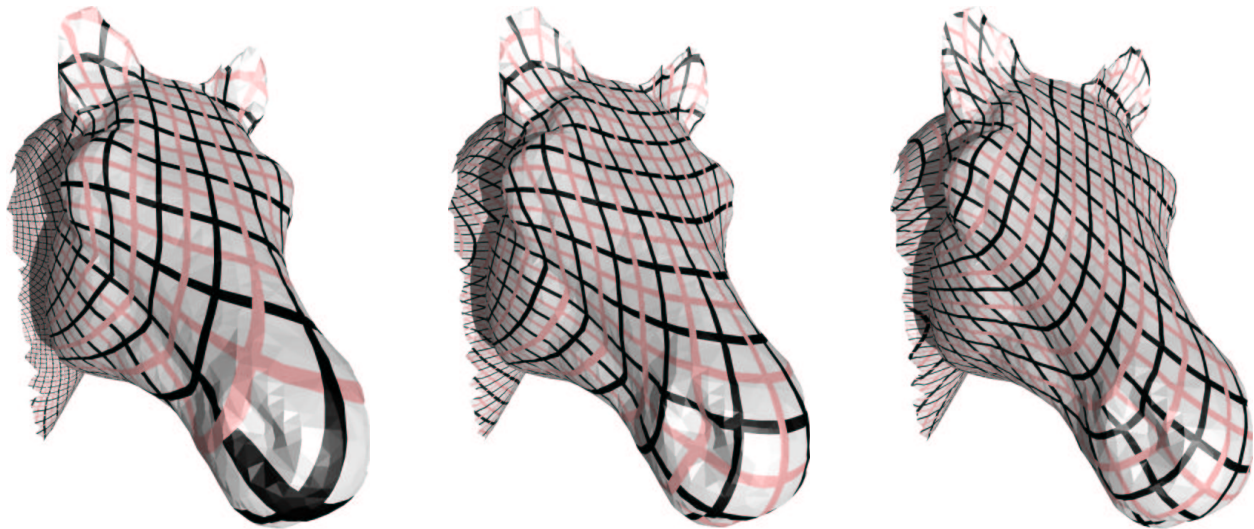


Figure 5: The horse head model parameterized using different angle/area preservation tradeoffs. From left to right, the values $\theta = 0.3, \theta = 1.0, \theta = 3.0$

5. CONCLUSION

In this paper we have proposed an energy functional that measures an isometric distortion of a parameterization. On surfaces for which no isometric parameterization exists, the functional weights global area and angle deformation in an intuitive and user-controlled way. We also showed, how the functional can be used to optimize for an uniform surface sampling.

Besides basic desirable properties, the functional can be continuously continued on degenerated parameterizations and does not depend on fixed boundary vertex texture coordinates which makes it possible to compute parameterizations without face flips and with optimal boundaries. Using conjugate gradient methods and hierarchical optimization we showed how the functional can be minimized efficiently.

In future works we would like to experiment with different edge collapse schedules in the generation of the splits sets during hierarchical optimization, to further speed up the computation of parameterizations. Furthermore we would like to address the numerical problems for higher values of θ .

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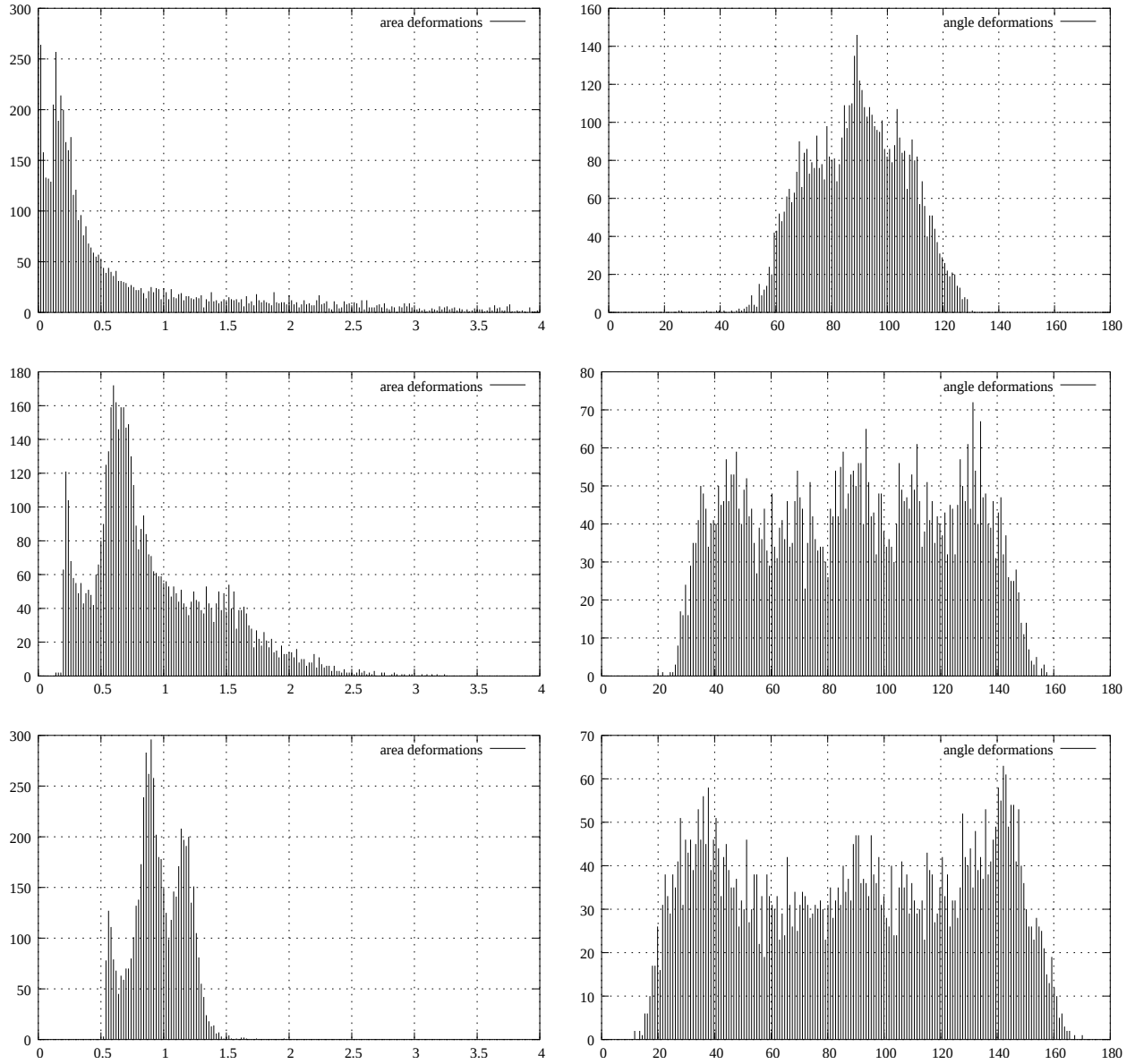


Figure 6: Per triangle distortions in area and angle for the horse head dataset for (from top to bottom) $\theta = 0.3$, $\theta = 1.0$ and $\theta = 3.0$, as shown in figure 5.

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