Supplementary Document on Non-Linearly Quantized Moment Shadow Maps

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ACM Reference format:
Christoph Peters. 2017. Supplementary Document on Non-Linearly Quantized Moment Shadow Maps. In *Proceedings of High Performance Graphics conference, Los Angeles, CA, USA, July 28-30, 2017 (HPG '17)*, 2 pages.
https://doi.org/10.1145/3105762.3105775

1 PROOFS

In the following we provide mathematical proof for various statements that are used within the paper.

1.1 Case Distinction for Shading

A convenient aspect of our non-linear representation of the moments is that y_1 and y_2 directly characterize the three different cases that occur in moment shadow mapping. In the following we assume $y_1 < y_2$ and $z_1 < z_2$. Then we claim that the shadow intensity is zero for $z_f \in (-\infty, y_1]$, w_1 for $z_f \in (y_1, y_2)$ and $w_1 + w_2 = 1 - w_0$ for $z_f \in (y_2, \infty)$. Here is a formalization of this statement.

Proposition 1. *For moments $b \in \mathbb{R}^5$ with $b_0 = 1$ and positive definite Hankel matrix $B(b)$ let $y_1, y_2 \in \mathbb{R}$ be the outputs of Algorithm 2. Let*

$$z_1, z_2 : \mathbb{R} \setminus \{y_1, y_2\} \to \mathbb{R}$$

describe the values computed by Algorithm 1 where $z_1(z_f) < z_2(z_f)$ for all $z_f \in \mathbb{R} \setminus \{y_1, y_2\}$. Let

$$\begin{aligned} N : \mathbb{R} \setminus \{y_1, y_2\} &\to \{0, 1, 2\} \\ z_f &\mapsto |\{z \in \{z_1(z_f), z_2(z_f)\} \mid z < z_f\}| \end{aligned}$$

be a map counting the weights that contribute to the shadow intensity. Then

$$N(z_f) = \begin{cases} 0 & \text{if } z_f \leq y_1, \\ 1 & \text{if } y_1 < z_f \leq y_2, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. As a first step, we prove that $z_1(z_f)$ and $z_2(z_f)$ are well-defined. According to the correctness proof of Algorithm 1 [Peters and Klein 2015, Proposition 3], these quantities are ill-defined if and only if $q_2 = 0$. Thus, we have to prove that $q_2 = 0$ if and only if $z_f \in \{y_1, y_2\}$. Let $e_2 := (0, 0, 1)^\mathsf{T}$ denote a canonical basis vector. Then by definition of q:

$$\begin{aligned} & q_2 = 0 \\ \Leftrightarrow\ & (\det B(b) \cdot e_2^\mathsf{T} \cdot B^{-1}(b)) \cdot \mathbf{b}(z_f) = 0 \\ \Leftrightarrow\ & (b_2 - b_1^2) \cdot z_f^2 + (b_1 \cdot b_2 - b_3) \cdot z_f + b_1 \cdot b_3 - b_2^2 = 0 \end{aligned}$$

This quadratic polynomial agrees with the polynomial in Algorithm 2 which has the roots y_1, y_2.

The functions $z_1(z_f)$ and $z_2(z_f)$ are compositions of continuous functions with no singularities besides y_1 and y_2. Therefore, they are continuous on $\mathbb{R} \setminus \{y_1, y_2\}$. Except for their order, they are fully characterized by the fact that the following matrix is an invertible diagonal matrix for all $z_f \in \mathbb{R} \setminus \{y_1, y_2\}$ [Peters and Klein 2015, Proposition 10]:

$$\begin{pmatrix} 1 & 1 & 1 \\ z_f & z_1(z_f) & z_2(z_f) \\ z_f^2 & z_1^2(z_f) & z_2^2(z_f) \end{pmatrix}^\mathsf{T} \cdot B^{-1}(b) \cdot \begin{pmatrix} 1 & 1 & 1 \\ z_f & z_1(z_f) & z_2(z_f) \\ z_f^2 & z_1^2(z_f) & z_2^2(z_f) \end{pmatrix}$$

It follows that the set $\{z_f, z_1(z_f), z_2(z_f)\}$ always has cardinality three because otherwise two matrices in this product would not be invertible. Furthermore, this set does not change if z_f is replaced by any other element of $\{z_f, z_1(z_f), z_2(z_f)\}$ because that only permutes rows and columns of the diagonal matrix in a symmetric fashion.

Suppose N has a discontinuity at $z \in \mathbb{R} \setminus \{y_1, y_2\}$. Then either the inequality $z_1(z_f) < z_f$ or $z_2(z_f) < z_f$ changes at $z_f = z$. Since both functions are continuous at z this implies either $z_1(z) = z$ or $z_2(z) = z$. Contradiction.

Finally, we note that a minimal choice of z_0 in $\{z_f, z_1(z_f), z_2(z_f)\}$ has to lead to $N(z_0) = 0$, while the largest choice leads to $N(z_0) = 2$ and the choice in between leads to $N(z_0) = 1$. Considering that N is continuous on $\mathbb{R} \setminus \{y_1, y_2\}$, this completes our proof. □

1.2 Bounds on y_1, y_2

Our quantization scheme exploits $y_1, y_2 \in [-1, 1]$. The proof of this statement is a simple consequence of Proposition 1.

Proposition 2. *Let Z be a distribution on $[-1, 1]$, let $b = \mathcal{E}_Z(\mathbf{b})$ and assume that $b_2 - b_1^2 > 0$. Let y_1, y_2 be the outputs of Algorithm 2 with $y_1 < y_2$. Then $y_1, y_2 \in [-1, 1]$.*

Proof. Suppose $y_1 < -1$. Consider $z_f \in \mathbb{R}$ with $y_1 < z_f < -1$ and $z_f < y_2$. By Proposition 1, the shadow intensity for z_f is given by

$$w_1 = \frac{1}{(1, z_1, z_1^2) \cdot B^{-1}(b) \cdot (1, z_1, z_1^2)^\mathsf{T}} > 0.$$

Obviously, this cannot be a lower bound to $Z(\mathbf{z} < z_f) = 0$. Contradiction.

Proving $y_2 \leq 1$ works analogously by considering $z_f \in \mathbb{R}$ with $1 < z_f < y_2$ and $y_1 < z_f$ and optimal upper bounds. □

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1.3 Bounds on ξ_4

Another inequality used by our quantization scheme is $\xi_4 \leq 0.25$.

Proposition 3. *Let Z be a distribution on $[-1, 1]$, let $b = \mathcal{E}_Z(\mathbf{b})$ and assume that $b_2 - b_1^2 > 0$. Let ξ_4 be the output of Algorithm 2. Then $\xi_4 \leq 0.25$ and this bound is sharp.*

Proof. For $z \in [-1, 1]$ we know $z^4 \leq z^2$ and thus

$$b_4 = \mathcal{E}_Z(\mathbf{z}^4) \leq \mathcal{E}_Z(\mathbf{z}^2) = b_2.$$

Now we consider the definition of ξ_4 from Algorithm 2:

$$\xi_4 = b_4 - b_2^2 - \frac{(b_3 - b_1 \cdot b_2)^2}{b_2 - b_1^2} \leq b_4 - b_2^2 \leq b_2 - b_2^2$$

Basic analysis shows that $b_2 - b_2^2$ takes its global maximum at $b_2 = 0.5$. Thus,

$$\xi_4 \leq b_2 - b_2^2 \leq 0.5 - 0.5^2 = 0.25.$$

To show that this upper bound is sharp, we consider the case

$$Z = 0.25 \cdot \delta_{-1} + 0.5 \cdot \delta_0 + 0.25 \cdot \delta_1.$$

Then

$$b = (1, 0, 0.5, 0, 0.5)$$

and therefore

$$\xi_4 = b_4 - b_2^2 - 0 = 0.5 - 0.5^2 = 0.25.$$

□

1.4 Scaling of ξ_4

For shading with a minimal number of operations we shift and scale the domain of depth values to achieve $y_1 = 0$ and $y_2 = 1$. This transform necessitates a change of ξ_4 that we derive below.

Proposition 4. *Let Z be a depth distribution on $\mathbb{R}$, let $b = \mathcal{E}_Z(\mathbf{b})$ and assume $b_2 - b_1^2 > 0$. Let $y_1, y_2, v_2, \xi_4 \in \mathbb{R}$ be the outputs of Algorithm 2 for input b. Let*

$$c := \frac{1}{y_2 - y_1}, \qquad d := -c \cdot y_1$$

be the scaling and shifting needed to normalize the depth values. For $j \in \{0, 1, 2, 3, 4\}$ let

$$b'_j := \mathcal{E}_Z((c \cdot \mathbf{z} + d)^j)$$

denote the moments of the scaled and shifted depth distribution. Let ξ'_4 be the output of Algorithm 2 for input b'. Then

$$\xi'_4 = c^4 \cdot \xi_4.$$

Proof. Let $S := (1 - v_2) \cdot \delta_{y_1} + v_2 \cdot \delta_{y_2}$ be the sparse distribution that reproduces the moments b_1, b_2, b_3 [Peters et al. 2017, Proposition 1]. Then by definition

$$\xi_4 = \mathcal{E}_Z(\mathbf{z}^4) - \mathcal{E}_S(\mathbf{z}^4).$$

Scaling and shifting just applies a linear transform to the moments [Peters et al. 2017, Equation (2)]. Thus, scaling and shifting both Z and S leads to a new pair of depth distributions where the first three moments agree. Since nothing changes about the sparsity of S, we obtain

$$\begin{aligned}
\xi'_4 &= \mathcal{E}_Z((c \cdot \mathbf{z} + d)^4) - \mathcal{E}_S((c \cdot \mathbf{z} + d)^4) \\
&= \mathcal{E}_Z\left(\sum_{j=0}^{4} \binom{4}{j} \cdot c^j \cdot \mathbf{z}^j \cdot d^{4-j}\right) - \mathcal{E}_S\left(\sum_{j=0}^{4} \binom{4}{j} \cdot c^j \cdot \mathbf{z}^j \cdot d^{4-j}\right) \\
&= \sum_{j=0}^{4} \binom{4}{j} \cdot c^j \cdot (\mathcal{E}_Z(\mathbf{z}^j) - \mathcal{E}_S(\mathbf{z}^j)) \cdot d^{4-j}.
\end{aligned}$$

In this sum, the terms for $j \in \{0, 1, 2, 3\}$ vanish because the corresponding moments of S and Z agree. What remains is

$$\xi'_4 = \binom{4}{4} \cdot c^4 \cdot (\mathcal{E}_Z(\mathbf{z}^4) - \mathcal{E}_S(\mathbf{z}^4)) \cdot d^0 = c^4 \cdot \xi_4.$$

□

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