Symbolic Equilibrium Point Analysis in Parameterized Polynomial Vector Fields

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Abstract. It is well known that various questions of stability of polynomial vectors fields can be reduced to quantifier elimination problems on real closed fields. More recently we have shown that also the parametric question of the occurrence of Hopf bifurcations can be decided by quantifier elimination. The combination of general purpose quantifier elimination systems has been sufficient to solve some of the occurring quantifier elimination problems but did not succeed for many others (on current computers). For the common case of equilibrium points with nonzero Jacobian determinant we will show that there is a computationally well suited description that can serve as an infrastructure for more efficient methods.

1 Introduction

Systems of ordinary differential equations are one of the most common mathematical structures used to model processes in the natural sciences. In general, these models lead to nonlinear systems which depend on parameters. Depending on the parameters their behavior might change dramatically. So the development of symbolic methods for their study is an important topic.

During the last decade many advances have been made for the symbolic study of differential equations [14, 13] using techniques such as Lie-symmetry methods [12, 17, 8] or differential Galois theory [15, 16]. This work is mainly aimed towards the symbolic solutions of systems and many important examples are not solvable in symbolic form so that these techniques are not applicable.

Nevertheless, very often only the *qualitative behavior* of a system of differential equations in dependency on the parameters is of interest. Also in this respect great advances have been obtained recently, e.g. for various questions of stability, such as the ones of some numerical integration schemes [9] or in connection with control theory [10].

That work uses the powerful technique of *quantifier elimination on real closed fields* [18], to which the questions on the differential equations are reduced—in the common case that the corresponding vector field is a polynomial system in the variables and parameters.

In our previous work [2] we studied *Hopf bifurcations* of parameterized polynomial vector fields and showed that the question of detecting such bifurcations can be reduced to a quantifier elimination problem. More precisely, we have proven that a system undergoes a Hopf bifurcation, at an equilibrium point $(\underline{u}, \underline{x})$, with empty unstable manifold if and only if

$$a_n(\underline{u}, \underline{x}) > 0, \Delta_{n-1}(\underline{u}, \underline{x}) = 0, \Delta_{n-2}(\underline{u}, \underline{x}) > 0, \ldots, \Delta_1(\underline{u}, \underline{x}) > 0 \quad ,$$

where a_n is, up to $(-1)^n$, the determinant of the Jacobian matrix $D(f)$ and the Δ_i's are the Hurwitz determinants associated to the characteristic polynomial of $D(f)$. For details on the notation, we refer to Sec. 2. This leads to the following first order formula:

$$\exists x \; (f(u,x) = 0, a_n(u,x) > 0, \Delta_{n-1}(u,x) = 0, \Delta_{n-2}(u,x) > 0 \ldots, \Delta_1(u,x) > 0) \tag{1}$$

In order to eliminate quantifiers from this formula one can use some existing quantifier elimination software as a black box. However, due to the particular nature of the formulas to be handled it is potentially much more efficient to integrate the quantifier elimination step into the general problem solution procedure, and to develop specialized algorithms for eliminating quantifiers from the first order formulas that arise from our analysis of the problem. As for many examples the existing quantifier elimination packages did not return a result within some days of computation time, cf. [2], such specialized methods are of great practical importance.

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1.1 Contributions of the present work

The way we deal with this difficulty in the present work can be summarized as follows: as the questions we ask on vector fields concern their equilibrium points, one expects that the obtained quantified formulas will involve the system of equations $f(u,x) = 0$. On the other hand, in the formula (1), and in many others arising from differential equations analysis, it is easy to see that the equilibrium points of interest have a nonzero Jacobian determinant. It is therefore natural to undertake a study of the solutions of the system $f(u,x) = 0$ that have a nonzero Jacobian determinant. As we shall see in Sec. 5.1, it turns out that such solutions have a nice description. More precisely, we will show that the set of parameters u can be partitioned into constructibles (i.e. sets given as solutions of polynomial equations and inequations) such that over any constructible the solutions of $f(u,x) = 0$ with $J(f)(u,x) \neq 0$ are given by a system of the form

$$p(u,w) = 0, \quad x_1 = c(u)^{-1} q_1(u,w), \quad \dots, \quad x_n = c(u)^{-1} q_n(u,w)$$

where c, p and the q_i's are polynomials depending on the constructible. Notice here that for any α in the given constructible the solutions of $f(\alpha,x) = 0$ are rational functions ($c(\alpha)^{-1} q_i(\alpha,w)$) in terms of the roots of a univariate polynomial ($p(\alpha,w)$). Such a representation is usually called a *Rational Univariate Representation*, see e.g. [5].

Once the partition into contructibles is completed—together with a rational univariate representation for each constructible—we use it to reduce the first order formulas into formulas with only one quantifier. Then we can apply any quantifier elimination software onto the obtained formulas.

1.2 Outline of the paper

This paper is divided into two different parts. The first one is aimed towards reducing different questions on equilibrium points of polynomial vector fields—such as stability and bifurcations detection—to quantifier elimination problems. The topic of the second part is a representation of the equilibrium points which is well suited for performing quantifier elimination.

In Sec. 2 we give a brief review on vector fields and stability criterions of equilibrium points. In Sec. 3 we give a link between Hurwitz determinants and subresultants theory. We then give explicit algebraic criterions for detecting the presence of symmetric roots, with respect to the origin of coordinates, and also the number of symmetric pairs for a given polynomial. We also study the behavior of Hurwitz determinants in the presence of symmetric roots. In Sec. 4 we use the obtained results to show how one can reduce various questions on equilibrium points of polynomial vector fields to quantifier elimination problems. Sec. 5 is devoted to the study of equilibrium points with nonzero Jacobian determinant. We give the main theoretical results which allow us to describe such points by using the concept of rational univariate representation. We then illustrate the usefulness of the given results with an example in Sec. 6.

2 Preliminaries

Let $f(u,x) = (f_1, \dots, f_n)$ be a parameterized vector field, where $f_i \in \mathbb{R}[u,x]$ are polynomials, $x = (x_1, \dots, x_n)$ is a list of variables and $u = (u_1, \dots, u_k)$ is a list of parameters. Let us consider the autonomous ordinary differential system

$$\dot{x} = f(u,x)$$

and by $\Phi_t(u,x)$ let us denote the flow generated by the vector field f. A good place to start the study of the nonlinear system $\dot{x} = f(u,x)$ is to find its equilibrium points, which are given by the equation

$$f(u,x) = 0.$$

If $\underline{u} \in \mathbb{R}^k$ and $(\underline{u}, \underline{x})$ is an equilibrium point of the specialized nonlinear system $\dot{x} = f(\underline{u}, x)$, the study of the behavior of the flow $\Phi_t(\underline{u}, x)$ when starting near the equilibrium point $(\underline{u}, \underline{x})$ is classically done using the linear system

$$\dot{\zeta} = D(f)(\underline{u}, \underline{x}) \cdot \zeta$$

where $D(f)(\underline{u}, \underline{x})$ is the Jacobian matrix of the vector field $f(\underline{u}, x)$ at the point $\underline{x}$. The flow generated by this linear system is then $e^{tD(f)(\underline{u},\underline{x})} \cdot \zeta = D(\Phi_t)(\underline{u}, \underline{x}) \cdot \zeta$.

A fundamental result due to Hartman and Gröbman (see e.g. [1, 6]) states that in the case of *hyperbolic* equilibrium point, i. e. the matrix $D(f)(\underline{u}, \underline{x})$ has no eigenvalue with zero real part, the nonlinear flow has the same topological behavior near the equilibrium point $(\underline{u}, \underline{x})$ as the linear flow near the origin 0. In particular, the nonlinear flow $\Phi_t(u, x)$ is asymptotically stable near the equilibrium point $(\underline{u}, \underline{x})$ if and only if all the eigenvalues of the matrix $D(f)(\underline{u}, \underline{x})$ have negative real part. According to the well known *Routh-Hurwitz criterion*, see e. g. [3, 9], this last condition is equivalent to the signs conjunction

$$\Delta_1(\underline{u}, \underline{x}) > 0, \ldots, \Delta_n(\underline{u}, \underline{x}) > 0 \quad ,$$

where the $\Delta_i(\underline{u}, \underline{x})$'s are the Hurwitz determinants associated to the characteristic polynomial of the matrix $D(f)(\underline{u}, \underline{x})$.

As the nonlinear system $\dot{x} = f(u, x)$ is parameterized, a natural question is to ask for which values $\underline{u}$ of the parameter u the specialized system $\dot{x} = f(\underline{u}, x)$ is asymptotically stable near all its equilibrium points. This can be symbolically expressed by the first order formula

$$\forall x \; (f(u, x) = 0 \Rightarrow \Delta_1(u, x) > 0, \ldots, \Delta_n(u, x) > 0) \quad . \tag{2}$$

One can also ask for which values $\underline{u}$ of the parameter u the system $\dot{x} = f(\underline{u}, x)$ is asymptotically stable near at least one of its equilibrium points. That is

$$\exists x \; (f(u, x) = 0, \Delta_1(u, x) > 0, \ldots, \Delta_n(u, x) > 0) \quad . \tag{3}$$

These questions, as many others, are thus reduced to quantifier elimination problems for first order formulas in the language of real closed fields.

2.1 Notations

In all the rest we will denote by $\mathbb{K}$ a commutative field of characteristic zero and by $\overline{\mathbb{K}}$ its algebraic closure. Let $\mathbb{A}$ be an affine ring over $\mathbb{K}$, i.e. a finitely generated $\mathbb{K}$-algebra $\mathbb{K}[u_1, \ldots, u_k]/\mathcal{J} = \mathbb{K}[u]/\mathcal{J}$. By parameterized polynomial vector field $f(x)$ with coefficients in $\mathbb{A}$, or often a vector field over $\mathbb{A}[x]$, we mean a list $f = (f_1(x), \ldots, f_n(x))$ where $f_i \in \mathbb{A}[x]$, and $x = (x_1, \ldots, x_n)$ is a list of variables. The ideal generated by the f_i's is denoted by $\mathcal{I}(f)$, the Jacobian matrix of f is denoted by $D(f)$ and the determinant of $D(f)$ is denoted by $J(f)$.

Localization ring If $\mathbb{A}$ is a commutative ring with unit and M is a nonempty subset of $\mathbb{A} \setminus \{0\}$ which is stable under multiplication, then we denote by $\mathbb{A}_M$ the localization ring of $\mathbb{A}$ with respect to M. When M is generated by a single element c, i.e. $M = \{c^n \; , \; n \in \mathbb{N}\}$ (resp. $M = \mathbb{A} \setminus \mathcal{P}$ where $\mathcal{P}$ is a prime ideal of $\mathbb{A}$) we use the notation $\mathbb{A}_c$ (resp. $\mathbb{A}_{\mathcal{P}}$) for short instead of $\mathbb{A}_M$.

Monomial semigroup We denote by $\mathbb{M} = \{x^\alpha \; ; \; \alpha \in \mathbb{N}^n\}$ the multiplicative semigroup generated by the indeterminates $x_1, \ldots, x_n$. By an admissible order of $\mathbb{M}$ we mean a total order relation $\preceq$ on $\mathbb{M}$ which is compatible with multiplication.

Leading monomial and leading term For a given polynomial $p = \sum_\alpha a_\alpha x^\alpha$ in $\mathbb{A}[x]$ we define the leading monomial $\mathrm{Lm}(p, \preceq)$ of p with respect to $\preceq$ to be x^β where x^β is the greatest monomial, with respect to $\preceq$, among the x^α's such that $a_\alpha \neq 0$. The leading term of p with respect to $\preceq$ is $\mathrm{Lt}(p, \preceq) = a_\alpha x^\alpha$ with $x^\alpha = \mathrm{Lm}(p, \preceq)$.

Initial ideal If $\mathcal{I}$ is an ideal of $\mathbb{A}[x]$ the ideal $\mathrm{Lt}(\mathcal{I}, \preceq)$ generated by the leading terms of the polynomials in $\mathcal{I}$ is called the *initial ideal* of $\mathcal{I}$. We also define the residue set of $\mathcal{I}$ with respect to $\preceq$ to be

$$\Gamma(\mathcal{I}, \preceq) = \{x^\alpha \in \mathbb{M} \; ; \; \forall p \in \mathcal{I} \;\; \mathrm{Lm}(p, \preceq) \neq x^\alpha\}.$$

3 Spectral analysis at the equilibrium points

By spectral analysis we mainly mean positioning, for a given equilibrium point, the eigenvalues of its Jacobian matrix with respect to the imaginary axis. Such information is usually quantified by computing the numbers of eigenvalues with positive, negative or zero real part.

Most of the results of this section are valid for arbitrary commutative fields of characteristic zero. Some others, specially those concerning spectrum positioning, require and order structure on the field. We shall formulate these results in the general setting of real closed fields. In the sequel we let $\mathbf{R}$ be a real closed field and $\mathbf{C} = \mathbf{R}(i)$ its algebraic closure.

Let $\chi(z) \in \mathbb{A}[z]$ be a polynomial of degree n (typically $\chi(z)$ is the characteristic polynomial of a square matrix of order n) and let us write

$$\chi(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n \quad .$$

The square matrix H of order n defined by

$$H = \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & \ldots \\ a_0 & a_2 & a_4 & \ldots & \ldots \\ 0 & a_1 & a_3 & a_5 & \ldots \\ 0 & a_0 & a_2 & a_4 & \ldots \\ & & & & \ddots \end{pmatrix}$$

is called the *Hurwitz matrix* of the polynomial $\chi(z)$. The i-th order principal minor of the matrix H is called the i-th *Hurwitz determinant* of the matrix H and is denoted by Δ_i. When the considered ring is a real closed field with order $\leq$ and $a_0 > 0$ then the well known Routh-Hurwitz criterion states that the polynomial χ has all its roots in the left half-plane (i.e. with negative real part) if and only if its Hurwitz determinants satisfy the sign conditions

$$\Delta_1 > 0, \ldots, \Delta_n > 0 \quad .$$

3.1 Hurwitz determinants as principal subresultant coefficients

In the sequel we shall use the Hurwitz determinants to give a criterion for the polynomial χ to have k pairs of symmetric roots with respect to the origin of the plane. We shall also give a criterion for the polynomial χ to have all its roots in the left half-plane except 2 roots $i\omega$ and $-i\omega$ which are in the imaginary axis. For this, we shall first express the Hurwitz determinants in terms of the principal subresultant coefficients of a pair of polynomials which are related to the polynomial $\chi(z)$. The proofs of the results we give in this section can be found in [2]

Definition 1. *Let $P, Q \in \mathbb{A}[y]$ be two polynomials,*

$$P = \sum_{k=0}^{p} a_k y^k \qquad Q = \sum_{k=0}^{q} b_k y^k$$

with $\deg(P) \leq p$ and $\deg(Q) \leq q$.

If $i \in \{0, \ldots, \min(p, q) - 1\}$ we define the subresultant polynomial associated to P, p and Q, q of index i as follows:

$$\mathrm{Sr}_i(P, p, Q, q) = \sum_{j=0}^{i} d_j^i y^j$$

where every d_j^i is the determinant of the matrix built with the columns $1, 2, \ldots, p+q-2i-1$ and $p+q-i-j$ in the following matrix:

$$M_i \; \overbrace{\begin{pmatrix} a_p & \ldots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_p & \ldots & a_0 \\ b_q & \ldots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_q & \ldots & b_0 \end{pmatrix}}^{p+q-i} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} q-i \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} p-i \end{matrix}$$

The determinant d_i^i is called i-th principal subresultant coefficient and denoted by

$$\mathrm{sr}_i(P, p, Q, q) \quad .$$

When no confusion arises, we shall write sr_i instead of $\mathrm{sr}_i(P, p, Q, q)$ and Sr_i instead of $\mathrm{Sr}_i(P, p, Q, q)$.

Let $\chi(z) \in \mathbb{A}[z]$ be a polynomial of degree n,

$$\chi(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n,$$

and let us write χ in the form

$$\chi(z) = \chi_1(z^2) + z\chi_2(z^2) \quad .$$

It is easy to see that $\deg(\chi_1) \leq [\frac{n}{2}]$ and $\deg(\chi_2) \leq [\frac{(n-1)}{2}]$, where $[.]$ denotes the floor function. Moreover, at least one of the two inequalities is an equality.

The Hurwitz determinants sequence of the polynomial $\chi(z)$ is in fact closely related to the principal subresultant sequence of the pair of polynomials χ_1 and χ_2. More precisely, one has the following theorem.

Theorem 1. *Let $\mathbb{A}$ be a commutative ring and $\chi \in \mathbb{A}[z]$ be a polynomial of degree n, and write*

$$\chi(z) = \chi_1(z^2) + z\chi_2(z^2).$$

Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be the Hurwitz determinants sequence of χ. Then for any $0 \leq i \leq [n/2] - 1$ one has :

$$\Delta_{n-2i-1} = \epsilon_i \mathrm{sr}_i(\chi_1, [n/2], \chi_2, [(n-1)/2]),$$

$$\Delta_{n-2i} = \epsilon'_i \mathrm{sr}_i(\chi_1, [n/2], y\chi_2, [(n-1)/2] + 1),$$

where $\epsilon'_i = (-1)^{\frac{([(n+1)/2]-i)([(n+1)/2]-i-1)}{2}}$ and $\epsilon'_i = (-1)^{\frac{([(n+1)/2]-i)([(n+1)/2]-i+1)}{2}}$.

As consequence of Theorem 1 we have the following algebraic criterion to detect the presence of symmetric roots for a given polynomial.

Corollary 1. *Let $\mathbb{K}$ be a commutative field of characteristic zero and $\chi(z) \in \mathbb{K}[z]$ be a polynomial of degree n. Then χ has k pairs of symmetric roots z_j and $-z_j$ if and only if*

$$\Delta_{n-1} = 0, \ldots, \Delta_{n-2k+1} = 0, \Delta_{n-2k-1} \neq 0 \quad .$$

3.2 Hurwitz determinants in the case of symmetric roots

We turn now to investigate another aspect of the Hurwitz determinants, namely the behavior of the Δ_i's when we add to the roots of the polynomial χ some pairs of symmetric points z_j and $-z_j$ of the plane.

Theorem 2. *Let $\chi(z), R(z) \in \mathbb{A}[z]$ be polynomials with $\deg(\chi) = n$ and $\deg(R) = r$. Let*

$$\chi^\star(z) = \chi(z)R(z^2) \quad .$$

If $\Delta^\star_i$ is the Hurwitz determinant of order i of the polynomial $\chi^\star(z)$ then

$$\begin{cases} \Delta_i = \Delta^\star_i & \text{for } i = 1, \ldots, n, \\ \Delta^\star_i = 0 & \text{for } i = n+1, \ldots, n+r \end{cases}$$

Remark 1. If we add to the roots of the polynomial χ the point 0 as root with multiplicity k, i.e. we take $\chi^\star(z) = \chi(z)z^k$, then we obtain the same conclusion as in Theorem 2.

4 From spectral analysis to semi-algebraic descriptions

In this section we explain, through some examples of well known bifurcations, how the tools developed in Sec. 3 can be used to produce first-order formulas describing any given kind of bifurcation.

4.1 The case of Hopf bifurcations

We are now able to give a semi-algebraic description of the set of real coefficients polynomials of a given degree which have one pair of roots, $i\omega$ and $-i\omega$, in the imaginary axis and no other root with zero real part.

Theorem 3. *Let $\chi(z) \in \mathbf{R}[z]$ be a degree n polynomial and write*

$$\chi(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n = \chi_1(z^2) + z\chi_2(z^2)$$

with $a_0 > 0$. Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be the Hurwitz determinants sequence of χ. Then $\chi(z)$ has a pair of distinct roots, $i\omega$ and $-i\omega$, in the imaginary axis and no other root with zero real part if and only if

$$\Delta_{n-1} = 0, a_n \Delta_{n-2} \Delta_{n-3} > 0 \quad .$$

As consequence of Theorems 1 and 2 we also get a nice semi-algebraic description of the set of real coefficients polynomials of a given degree which have all their roots in the left half-plane except one pair, $i\omega$ and $-i\omega$, in the imaginary axis.

Theorem 4. *Let $\chi(z) \in \mathbf{R}[z]$ be a degree n polynomial and write*

$$\chi(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n = \chi_1(z^2) + z\chi_2(z^2)$$

with $a_0 > 0$. Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be the Hurwitz determinants sequence of χ. Then $\chi(z)$ has a pair of distinct roots, $i\omega$ and $-i\omega$, in the imaginary axis and all the other roots in the left half–plane if and only if

$$a_n > 0, \Delta_{n-1} = 0, \Delta_{n-2} > 0, \ldots, \Delta_1 > 0 \quad .$$

As consequence of the last theorem, one can describe the parameters set for which a Hopf bifurcation with empty unstable manifold occurs by the following first-order formula :

$$\exists x \; (f(u,x) = 0, a_n(u,x) > 0, \Delta_{n-1}(u,x) = 0, \Delta_{n-2}(u,x) > 0 \ldots, \Delta_1(u,x) > 0)$$

4.2 The general case

Bifurcations that may occur in a given parameter dependent polynomial vector field are combinations of the two following cases:

- The value 0 is an eigenvalue of multiplicity m: This can be detected by checking whether the m first coefficients of the characteristic polynomial are zero, while the coefficient of degree m is not zero.
- There are k pairs of eigenvalues of the form $\pm i\omega$, where ω is a positive real number: this can be detected by checking whether the polynomials χ_1 and χ_2 have a gcd χ_3 of degree $\geq k$ with k negative roots. To get a semi-algebraic description in this case, one first has to introduce a disjunction depending on whether the degree of χ_3 is $k, k+1, \ldots$ (this can be achieved by using Corollary 1). Then one needs to express the fact that χ_3 has exactly k negative roots for each potential degree. This can be done e.g. by using the Sturm-Habicht sequence of χ_3 and its first derivative (obtained from the subresultant sequence by signs modifications so that the root counting problems can be handled).

4.3 Comparison with existing algebraic methods

In [7] the authors basically deal with the problem of detecting Hopf bifurcation in parameter dependent vector fields. Even though their concern is mainly aimed towards numerical algorithms, the method they use is similar to ours in so far as they use the subresultant sequence of χ_1 and χ_2. However, they only establish similar results to the ones given in Corollary 1 and Theorem 3.

Another main feature of our method, which is due to Theorems 1 and 2, is that we can address at the same time and with the same tool, namely Hurwitz determinants, two questions: The problem of detecting symmetric roots but also the problem of positioning the rest of the roots with respect to the imaginary axis. There are no analogous results to Theorem 1 and Theorem 2 in [7], and therefore many important questions on equilibrium points, such as bifurcations with empty unstable manifold, cannot be detected by their method.

5 Equilibrium points with nonzero Jacobian determinant

In the rest of this section we shall be concerned with equilibrium points of a vector field $f(u,x)$ with nonzero Jacobian determinant. More precisely, we shall investigate the structure of the set

$$\mathcal{V}(f) = \{(\underline{u}, \underline{x}) \in \overline{\mathbb{K}}^{k+n} \; / \; f_1(\underline{u}, \underline{x}) = 0, \ldots, f_n(\underline{u}, \underline{x}) = 0, \; J(f)(\underline{u}, \underline{x}) \neq 0\},$$

and its projection

$$\mathcal{W}(f) = \{\underline{u} \in \overline{\mathbb{K}}^k \mid \exists \, \underline{x} \in \overline{\mathbb{K}}^n \; , \; (\underline{u}, \underline{x}) \in \mathcal{V}(f)\}.$$

This undertaking is motivated by the following easy fact: the equilibrium points $(\underline{u}, \underline{x})$ of the system which are asymptotically stable (formulas (2) and (3)), as well as those undergoing a Hopf bifurcation (formula (1)) have a nonsingular Jacobian matrix so that they belong to $\mathcal{V}(f)$.

Our purpose in this section will be the construction of a list

$$[\mathcal{J}_i, c_i(u), p_i(u,w), q_{i,1}(u,w), \ldots, q_{i,n}(u,w) \; ; \; i = 1, \ldots, m],$$

where $\mathcal{J}_i = \mathcal{I}(a_{i,1}, \ldots, a_{i,r_i})$ is an ideal of $\mathbb{K}[u]$ and $c_i(u)$, $p_i(u,w)$ and the $q_{i,j}$'s are polynomials, satisfying the following properties:

(i) The sequence $(\mathcal{J}_i)_{1\leq i\leq m}$ is increasing, $\mathcal{J}_1 = \{0\}$ and $c_i \in \mathcal{J}_{j+1}$;
(ii) the constructibles $\mathcal{C}_i = \{\alpha \in \overline{\mathbb{K}}^k \ ; \ a_{i,1}(\alpha) = \ldots a_{i,r_i}(\alpha) = 0 \ , \ c_i(\alpha) \neq 0\}$ form a partition of $\mathcal{W}(f)$;
(iii) for any $i = 1, \ldots, m$ and any $\alpha \in \mathcal{C}_i$ the solutions of the system $f(\alpha, x) = 0$ are given by the rational univariate representation

$$p_i(\alpha, w) = 0, \quad x_1 = c_i(\alpha)^{-1} q_{i,1}(\alpha, w), \quad \ldots, \quad x_n = c_i(\alpha)^{-1} q_{i,n}(\alpha, w).$$

We need the following well-known lemma, see e.g. [19].

Lemma 1. *Let $\mathbb{A}$ be an integral affine ring of transcendence degree k over a field $\mathbb{K}$ and f be a vector field over $\mathbb{A}[x]$. Then the following assertions hold:*

(i) Any minimal prime divisor $\mathcal{P}$ of $\mathcal{I}(f)$ such that $J(f) \notin \mathcal{P}$ is of dimension k and $\mathbb{A} \cap \mathcal{P} = \{0\}$.
(ii) If $J(f)$ is not a zero divisor in $\mathbb{A}[x]/\mathcal{I}(f)$ then the ideal $\mathcal{I}(f)$ is radical equidimensional of dimension k. In particular, $\mathbb{A} \cap \mathcal{I}(f) = \{0\}$ and $\mathbb{A}[x_i] \cap \mathcal{I}(f) \neq \{0\}$ for any $i = 1, \ldots, n$.

Remark 2. If $\mathbb{A}$ is a field and f is a vector field over $\mathbb{A}[x]$ such that $J(f)$ is a unit in $\mathbb{A}[x]/\mathcal{I}(f)$ then the ideal $\mathcal{I}(f)$ is radical and zero-dimensional. In particular it has finitely many zeros, each one of multiplicity 1.

Let us now state the main procedure to be iterated in order to obtain the required description of the zeros of $\mathcal{I}(f)$ that have a nonzero Jacobian determinant.

Theorem 5. *Let $\mathbb{A} = \mathbb{K}[u]/\mathcal{J}$ be an affine ring, f be a vector field over $\mathbb{A}$ such that $J(f)$ is a unit in $\mathbb{A}[x]/\mathcal{I}(f)$, and assume that $\mathcal{I}(f) \cap \mathbb{A} = \{0\}$. Then there exist a polynomial $c(u) \in \mathbb{K}[u]$ and polynomials*

$$p(u, w), q_1(u, w), \ldots, q_n(u, w) \in \mathbb{K}[u, w]$$

satisfying the following properties:

(i) The polynomial $c(u)$ is non-nilpotent in $\mathbb{A}$.
(ii) The leading coefficient, with respect to w, of the polynomial $p(u, w)$ is a unit in the localization ring $\mathbb{A}_c$.
(iii) For any zero α of $\mathcal{J}$ in $\overline{\mathbb{K}}^k$ such that $c(\alpha) \neq 0$ the zeros in $\overline{\mathbb{K}}^n$ of the system $f(\alpha, x) = 0$ are given by the system

$$p(\alpha, w) = 0, \quad x_1 = c(\alpha)^{-1} q_1(\alpha, w), \quad \ldots, \quad x_n = c(\alpha)^{-1} q_n(\alpha, w) \quad .$$

Proof. Before giving the details of the proof of the theorem, we will give some explanations concerning the meaning of the claimed properties should be given. This should help to clarify what will be done in the proof.

For our purpose the ring $\mathbb{A}[x]/\mathcal{I}(f)$ represents—undoing multiplicities—the zeros (α, β) in $\overline{\mathbb{K}}^{k+n}$ of the ideal generated over $\mathbb{K}[u, x]$ by $\mathcal{J}$ and $\mathcal{I}(f)$, and our goal will be to find a rational univariate representation of such points. In general, it is not possible to find a uniform rational univariate representation for all these points. The role of $c(u)$ will be precisely the selection of points—by the constraint $c(u) \neq 0$—, for which such a representation is possible. The constraint $c(u) \neq 0$ selects a nonempty set if and only if $c(u)$ is non-nilpotent in $\mathbb{A}$. This is the reason why property (i) is required. By property (ii) we ensure that for any solution α of $\mathcal{J}$ such that $c(\alpha) \neq 0$ the number of solutions of the system $f(\alpha, x) = 0$ is the same. The purpose of property (iii) is that the points with $c(\alpha) \neq 0$ have a uniform rational univariate representation.

Let us now give the details of the proof. In fact, the construction of the polynomials

$$c(u), p(u, w), q_1(u, w), \ldots, q_n(u, w)$$

will be achieved in several steps, and we need to construct several intermediate objects to achieve this goal. Each one of these steps can be seen as a subroutine of an algorithmic procedure that allows us to construct the required polynomials.

First step : Let $\preceq$ be an admissible order on $\mathbb{M}$ and let $B = \{a_{\alpha^{(1)}} x^{\alpha^{(1)}}, \ldots, a_{\alpha^{(r)}} x^{\alpha^{(r)}}\}$ be a minimal generating system of the initial ideal $\mathrm{Lt}(\mathcal{I}(f), \preceq)$.

- If all the a_α's are non-nilpotent in $\mathbb{A}$ then we let $a = \prod_\alpha a_\alpha$ and this ends the current step.
- If some of the a_α's are nilpotent in $\mathbb{A}$ then we let $\mathcal{J}_1$ be the ideal of $\mathbb{K}[u]$ generated by $\mathcal{J}$ and the nilpotents among the a_α's. We then replace $\mathbb{A}$ by $\mathbb{A}_1 = \mathbb{K}[u]/\mathcal{J}_1$ and go back to compute a minimal generating system of the initial ideal $\mathrm{Lt}(\mathbb{A}_1[x]\mathcal{I}(f), \preceq)$. Let us notice here that $\mathcal{J} \subseteq \mathcal{J}_1 \subseteq \sqrt{\mathcal{J}}$ so that $\mathcal{J}$ and $\mathcal{J}_1$ have the same zeros in $\overline{\mathbb{K}}^k$.

Since $\mathbb{K}[u]$ is Noetherian, after a finite number of iterations of the previous loop we will get an ideal $\mathcal{J}_s$ (with $\mathbb{A}_s = \mathbb{K}[u]/\mathcal{J}_s$) such that $\mathcal{J} \subseteq \mathcal{J}_s \subseteq \sqrt{\mathcal{J}}$ and the initial ideal $\mathrm{Lt}(\mathbb{A}_s[x]\mathcal{I}(f), \preceq)$ has a minimal generating system $B = \{a_{\alpha^{(1)}} x^{\alpha^{(1)}}, \ldots, a_{\alpha^{(r)}} x^{\alpha^{(r)}}\}$ with $a = \prod_\alpha a_\alpha$ non-nilpotent in $\mathbb{A}_s$. For seek of simplicity and without loss of generality we will suppose in the sequel that $\mathbb{A} = \mathbb{A}_s$.

Second step : Now we turn to the construction of the polynomials p and q_i's. Since a is non-nilpotent in $\mathbb{A}$ the localization ring $\mathbb{A}_a$ is not reduced to $\{0\}$. Moreover, the initial ideal $\mathrm{Lt}(\mathbb{A}_a[x]\mathcal{I}(f), \preceq)$ is generated by $\{x^{\alpha^{(1)}}, \ldots, x^{\alpha^{(r)}}\}$, and hence the ring $\mathbb{B} = \mathbb{A}_a[x]/\mathbb{A}_a[x]\mathcal{I}(f)$ is free of finite rank as $\mathbb{A}_a$-module.

Let $t = (t_1, \ldots, t_n)$ be a list a indeterminates. Then the rings $\mathbb{A}_a[t]$ and $\mathbb{B}$—viewed as $\mathbb{A}_a$-algebras—are linearly disjoint over $\mathbb{A}_a$. Thus, any basis of $\mathbb{B}$ over $\mathbb{A}_a$ is a basis of $\mathbb{B}[t]$ over $\mathbb{A}_a[t]$.

Let $v = t_1x_1 + \ldots + t_nx_n$ and let L_v be the $\mathbb{A}_a[t]$-endomorphism of the multiplication by v in $\mathbb{B}[t]$. Let $\chi(w)$ be the characteristic polynomial of L_v. This can be viewed as polynomial in $\mathbb{K}(u)[t, w]$, and multiplying it by a suitable power of a we get a polynomial $P(u, t, w)$ in $\mathbb{K}[u, t, w]$. Let us remark that

$$P(u, t, t_1x_1 + \ldots + t_nx_n) = 0 \tag{4}$$

in the ring $\mathbb{B}[t]$, and hence for any $\tau = (\tau_1, \ldots, \tau_n) \in \overline{\mathbb{K}}^n$ and any zero (α, β) of $\mathcal{I}(f)$ in $\overline{\mathbb{K}}^n$ such that $a(\alpha) \neq 0$ we have

$$P(\alpha, \tau, \tau_1\beta_1 + \ldots + \tau_nx_n) = 0 \quad . \tag{5}$$

Moreover, for any zero α of $\mathcal{J}$ such that $a(\alpha) \neq 0$ we have

$$P(\alpha, t, w) = a^r \prod_{i=1}^{N} (w - \beta_1^{(i)} t_1 - \ldots - \beta_n^{(i)} t_n) \quad , \tag{6}$$

where $r \geq 0$ and $\beta^{(1)}, \ldots, \beta^{(N)}$ are the zeros of the system $f(\alpha, x) = 0$.

Let $D(u, t)$ be the discriminant of P with respect to w. Then given a zero α of $\mathcal{J}$ such that $a(\alpha) \neq 0$ and following equation (6) we have

$$D(\alpha, t) = a(\alpha)^s \prod_{i \neq j} \sum_k (\beta_k^{(i)} - \beta_k^{(j)}) t_k$$

where s depends only on r and N. This proves in particular that $D(\alpha, t) \neq 0$ and as by product that $D(u, t)$ is non-nilpotent in $\mathbb{A}[t]$.

The polynomial $p(u, w)$ we are looking for will be obtained from $P(u, t, w)$ by a suitable specialization of the parameters list t. Before we show how this can be done, let us construct polynomials $Q_i(u, t, w)$ which will give $q_i(u, w)$ after specialization. For this let us write

$$D(u, t) = U(u, t, w)P(u, t, w) + V(u, t, w)\partial_w P(u, t, w)$$

and let $Q_i(u, t, w) = -V(u, t, w)\partial_{t_i}(u, t, w)$. Applying the differential operator ∂_{t_i} to equation (4) we get

$$\partial_{t_i} P(u, t, t_1x_1 + \ldots + t_nx_n) + x_i \partial_w P(u, t, t_1x_1 + \ldots + t_nx_n) = 0$$

in $\mathbb{B}[t]$, and multiplying both sides of this equality by $V(u, t, t_1x_1 + \ldots + t_nx_n)$ we get, after simplification,

$$D(u, t)x_i - Q_i(u, t, t_1x_1 + \ldots + t_nx_n) = 0 \tag{7}$$

in $\mathbb{B}[t]$. Now let α be any zero of $\mathcal{J}$ such that $a(\alpha) \neq 0$. Then according to equation (6) the roots of the polynomial $P(\alpha, t, w)$, when viewed as polynomial with coefficients in $\overline{\mathbb{K}}(t)$, are of the from $\gamma = \beta_1t_1 + \ldots + \beta_nt_n$, where $\beta = (\beta_1, \ldots, \beta_n)$ is a zero of the system $f(\alpha, x) = 0$. Moreover, according to equation (7), the β_i's are given by $\beta_i = D(\alpha, t)^{-1}Q_i(\alpha, t, \gamma)$.

This means that the system

$$P(u, t, w) = 0, \quad x_1 = D(u, t)^{-1}Q_1(u, t, w), \quad \ldots, \quad x_n = D(u, t)^{-1}Q_n(u, t, w)$$

is a rational univariate representation of the zeros (α, β) of $\mathcal{I}(f)$ such that $a(\alpha) \neq 0$. To obtain the desired rational univariate representation we just need to get rid of the parameters list t in the previous one. This will be achieved by suitably specializing t and excluding more points than the ones excluded by the condition $a(u) = 0$.

Let z be an indeterminate and let $D_1(u, z) = D(u, 1, z, \ldots, z^{n-1})$. Given a zero α of $\mathcal{J}$ such that $a(\alpha) \neq 0$ and $\beta^{(1)}, \ldots, \beta^{(N)}$ the zeros of the system $f(\alpha, x) = 0$ we have

$$D_1(\alpha, z) = a(\alpha)^s \prod_{i \neq j} \sum_k (\beta_k^{(i)} - \beta_k^{(j)}) z^{k-1} \neq 0$$

and hence $D_1(u,z)$ is non-nilpotent in $\mathbb{A}[z]$. Let h be the degree of $D_1(u,t)$ with respect to z and write $D_1(u,z) = c_h z^h + \ldots + c_1 z + c_0$. For any $k = 0, 1, \ldots, h$ let $e_k = D(u,k)$. Then we have

$$(e_0, \ldots, e_h)^T = M(c_0, \ldots, c_h)^T$$

where M is the Vandermonde matrix associated with $0, 1, \ldots, h$. Since $\mathbb{A}$ is of characteristic zero the matrix M is invertible and we have

$$(c_0, \ldots, c_h)^T = M^{-1}(e_0, \ldots, e_h)^T.$$

In particular, the c_k's are linear combinations of the e_k's. On the other hand, since $D_1(u,z)$ is non-nilpotent in $\mathbb{A}[z]$, at least one of its coefficients is non-nilpotent in $\mathbb{A}$ and hence at least one of the e_k's is non-nilpotent in $\mathbb{A}$, say e_{k_0}.

Now let $c(u) = D_1(u, k_0)$, $p(u,w) = P(u, 1, k_0, \ldots, k_0^{n-1}, w)$, and $q_i(u,w) = Q_i(u, 1, k_0, \ldots, k_0^{n-1}, w)$ (for $i = 1, \ldots n$). Then the constructed polynomials satisfy the claimed properties.

Remark 3. (i) A step by step analysis of the previous proof shows that all the involved objects can be computationally constructed, and hence it leads to an effective algorithm that allows to compute the required polynomials, namely $c(u)$, $p(u,w)$ and the $q_i(u,w)$'s. Moreover, the basic operation used, apart from the computation of characteristic polynomials, is nilpotency checking. This can for example be effectively carried out using Gröbner bases computations.

(ii) The fact that we have added n indeterminates $t_1, \ldots, t_n$ to compute p and the q_i's is not really necessary, and has been done only for seeking simplicity in the formulation of the q_i's. In fact, only one added indeterminate z suffices, provided that we pay the price of computing some traces of endomorphisms defined over $\mathbb{B}[z]$ (we refer to [5] for the details of such methods).

5.1 Description of equilibrium points with nonzero Jacobian determinant

Now we have all of the necessary material to state the main result of this section.

Theorem 6. *Let f be a vector field over $\mathbb{K}[u]$ such that $J(f)$ is a unit in the quotient ring $\mathbb{K}[u,x]/\mathcal{I}(f)$. Then there exists a list*

$$[\mathcal{J}_i, c_i(u), p_i(u,w), q_{i,1}(u,w), \ldots, q_{i,n}(u,w)\ ;\ i = 1, \ldots, m],$$

where $\mathcal{J}_i = \mathcal{I}(a_{i,1}, \ldots, a_{i,r_i})$ is an ideal of $\mathbb{K}[u]$ and $c_i(u)$, $p_i(u,w)$ and the $q_{i,j}$'s are polynomials, satisfying the following properties:

(i) the sequence $(\mathcal{J}_i)_{1 \le i \le m}$ is increasing and $\mathcal{J}_1 = \{0\}$,

(ii) the constructibles $\mathcal{C}_i = \{\alpha \in \overline{\mathbb{K}}^k\ ;\ a_{i,1}(\alpha) = \ldots a_{i,r_i}(\alpha) = 0\ ,\ c_i(\alpha) \neq 0\}$ form a partition of $\mathcal{W}(f)$,

(iii) for any $i = 1, \ldots, m$ and any $\alpha \in \mathcal{C}_i$ the solutions of the system $f(\alpha, x) = 0$ are given by the rational univariate representation

$$p_i(\alpha, w) = 0, \quad x_1 = c_i(\alpha)^{-1} q_{i,1}(\alpha, w), \quad \ldots, \quad x_n = c_i(\alpha)^{-1} q_{i,n}(\alpha, w) \quad .$$

Proof. The main idea consists in iterating the procedure of Theorem 5 until all the points of $\mathcal{W}(f)$ are filled. We apply the procedure to f viewed over $\mathbb{K}[u]$ to compute polynomials $c_1(u), p_1(u,w), q_{1,1}(u,w), \ldots, q_{1,n}(u,w)$ such that for any $\alpha \in \overline{\mathbb{K}}^k$ with $c_1(\alpha) \neq 0$ the zeros of the system $f(\alpha, x) = 0$ are given by the rational univariate representation

$$p_1(\alpha, w) = 0, \quad x_1 = c_1(\alpha)^{-1} q_{1,1}(\alpha, w), \quad \ldots, \quad x_n = c_1(\alpha)^{-1} q_{1,n}(\alpha, w) \quad .$$

Now we should deal with the points α such that $c_1(\alpha) = 0$,. For this purpose we let $\mathcal{J}_2 = \mathcal{I}(c_1(u))$ and apply the same theorem to the vector field f, which is now viewed over $(\mathbb{K}[u]/\mathcal{J}_2)[x]$. This allows us to construct polynomials $c_2(u)$, $p_2(u,w)$, $q_{2,1}(u,w)$,..., $q_{2,n}(u,w)$ such that for any zero α of $\mathcal{J}_1$ with $c_2(\alpha) \neq 0$ the zeros of the system $f(\alpha, x) = 0$ are given by the rational univariate representation

$$p_2(\alpha, w) = 0, \quad x_1 = c_2(\alpha)^{-1} q_{2,1}(\alpha, w), \quad \ldots, \quad x_n = c_2(\alpha)^{-1} q_{2,n}(\alpha, w).$$

Now we let $\mathcal{J}_3 = \mathcal{I}(\mathcal{J}_1, c_2(u))$ and continue in the same way with the vector field f viewed over $(\mathbb{K}[u]/\mathcal{J}_3)[x]$. Notice here that c_2 is non-nilpotent in $(\mathbb{K}[u]/\mathcal{J}_2)[x]$ and hence $\mathcal{J}_2 \subset \mathcal{J}_3$.

Continuing this way we construct the required sequence with the claimed properties. Since $\mathbb{K}[u]$ is Noetherian and the sequence $(\mathcal{J}_i)$ is increasing it will stop after a finite number of iterations, say m iterations. The fact that for any zero α of $\mathcal{I}(\mathcal{J}_m, c_m)$ the system $f(\alpha, x) = 0$ has no solution is a direct consequence of Theorem 5.

6 Computational examples

6.1 Using general purpose quantifier elimination systems

On the basis of the evolving software-component architecture described in [4, 20] we implemented the method stated in Sec. 4.1 in a combined system of Maple, REDLOG, and QEPCAD, cf. [2]. Using these "general purpose quantifier elimination systems" we were able to solve the parametric question on the existence of Hopf bifurcations.

Example 1. On of the examples given in [2] is the famous "Lorenz System", which is given by the following system of ODEs:

$$\begin{aligned}\dot{x}(t) &= \alpha\,(y(t) - x(t))\\ \dot{y}(t) &= r\,x(t) - y(t) - x(t)\,z(t)\\ \dot{z}(t) &= x(t)\,y(t) - \beta\,z(t)\end{aligned}$$

Applying our program described in [2] to the Lorenz system imposing positivity conditions on the parameters gave the following answer after some seconds of computation time:

$$\alpha^2 + \alpha\beta - \alpha r + 3\alpha + \beta r + r = 0 \;\wedge\; \alpha r - \alpha - \beta^2 - \beta \geq 0 \;\wedge\; 2\alpha - 1 \geq 0 \;\wedge\; \beta > 0$$

Thus we have found a simple closed from description involving three free parameters, which coincides (after some elementary transformation) with the result of a hand computation given in [6].

A system arising in epidemiology The following example is from [11]. In this research paper the investigation on the existence of Hopf bifurcations is an important part. The differential equations come from epidemiological models with varying population size and dose-dependent latency period.

Example 2. The following parameterized system of differential equations describes the so called SEIS[1] models of [11]

$$\begin{aligned}\dot{s}(t) &= b - b\,s(t) + \delta\,i(t) - (\beta - \alpha)\,s(t)\,i(t)\\ \dot{e}(t) &= -b\,e(t) + \beta\,s(t)\,i(t) + \alpha\,i(t)\,e(t) - \varepsilon\,e(t)\\ \dot{i}(t) &= -(b + \delta + \alpha)\,i(t) + \alpha\,i(t)^2 + \varepsilon\,e(t)\end{aligned}$$

In [11] it is proved that this system does not have a Hopf bifurcation for any parameter values for the epidemiological relevant cases: all parameters and variables are positive and $s(t) + e(t) + i(t) = 1$.

Using our previously developed software, the quantifier elimination programs did not succeed for the general system with 3 variables and 5 parameters within one day of computation time.

When specializing 4 of the 5 parameters with various values, the combination of REDLOG and QEPCAD returned the correct result, namely `false`, within some seconds of computation time.

6.2 Computing the constructibles

We have not implemented the method described in Sec. 5 yet. However, we will show on the example of the SEIS model that the method given Sec. 5 can be performed on systems, on which the general method has failed.

We will do the computation on the model with 2 parameters. We set $b = \alpha = \beta = 1$, and use x, y, z as names of the variables instead of s, i, e.

Thus let us consider the vector field $f = (p, q, r)$ with coefficients in $\mathbb{R}[\delta, \varepsilon]$ given by

$$\begin{aligned}p &= 1 - x + \delta\,y\\ q &= -z + yx + yz - \varepsilon\,z\\ r &= (2 + \delta)\,y + y^2 + \varepsilon\,z\end{aligned}$$

The first constructible we find is given by the constraint

$$\varepsilon\,\delta\;(\varepsilon + 1)\,(\delta + 1)\,(\varepsilon\,\delta + \varepsilon + 2 + \delta) \neq 0$$

[1] SEIS stands for susceptibles (S), which can become exposed (E), i. e. are infected but not yet infectious, which will become infectious (I), which then become susceptibles (S) again.

and the corresponding rational univariate representation

$$\begin{array}{l}
(-1+x)\,(-\delta+x-1)\left(x-\varepsilon\,\delta-\delta^2\varepsilon-1-\delta^2-2\,\delta\right)=0\\
y=\varepsilon\,v\,(\varepsilon+1)\,(\delta+1)\,(\varepsilon\,\delta+\varepsilon+2+\delta)\,(-1+x)\\
z=v\,(\varepsilon+1)\,(-1+x)\,(-7+3\,x-8\,\varepsilon^2v-v\varepsilon^4-5\,\delta^2-4\,\varepsilon\,v+2\,v\varepsilon^4\delta\,x+v\varepsilon^4\delta^2x\\
\quad-\delta^3\varepsilon-9\,\delta\quad-6\,\varepsilon\,\delta+4\,v\varepsilon\,x+4\,\varepsilon^3\delta^2vx-4\,\delta^2\varepsilon-4\,\varepsilon+\varepsilon\,x\delta-13\,\varepsilon^2\delta\,v\\
\quad-9\,\varepsilon^3\delta\,v-4\,\varepsilon^3\delta^2v-5\,\varepsilon^2\delta^2v\quad-6\,\varepsilon\,\delta\,v+5\,\varepsilon^2\delta^2vx+2\,\varepsilon\,\delta^2vx-\delta^3+9\,\varepsilon^3\delta\,vx\\
\quad+6\,\varepsilon\,\delta\,vx+13\,\varepsilon^2\delta\,vx+2\,x\varepsilon-2\,\varepsilon\,\delta^2v-2\,v\varepsilon^4\delta\quad-v\varepsilon^4\delta^2+v\varepsilon^4x\\
\quad+8\,\varepsilon^2vx+5\,\varepsilon^3vx-5\,\varepsilon^3v+x\delta)
\end{array}$$

where v stands for the inverse of $\varepsilon\,\delta\;(\varepsilon+1)\,(\delta+1)\,(\varepsilon\,\delta+\varepsilon+2+\delta)$. Here there was no need to introduce a new variable w (this means according to the notations of Theorem 6 that we take $t=0$). Now we turn to partition the algebraic set given by the equation

$$\varepsilon\,\delta\;(\varepsilon+1)\,(\delta+1)\,(\varepsilon\,\delta+\varepsilon+2+\delta)=0 \tag{8}$$

As this equation is presented in factored form it is easier to investigate each factor alone. For the equation $\delta+1=0$ and $\varepsilon+1=0$ we find the same representation

$$\begin{array}{l} x=1\\ y=0\\ z=0 \end{array}$$

For the equation $\varepsilon=0$ we find the representation

$$\begin{array}{l}
(-1+w)\left(w+3+3\,\delta^2+8\,\delta\right)=0\\
x=\dfrac{1}{3}(2\,v\delta\,w+4\,vw-w-2\,v\delta-4\,v+4)\\
y=-v\,(2+\delta)\,(-1+w)\\
z=\dfrac{1}{3}-(2\,v+1+v\delta)\,(-1+w)
\end{array}$$

where v is in this case the inverse of $(2+\delta)\,(3\,\delta+2)$. This gives the constructible defined by the constraints

$$\varepsilon=0,\;(2+\delta)\,(3\,\delta+2)\neq 0.$$

Before going back to the other factors of equation (8) we should see what happens for the algebraic set given by

$$\varepsilon=0,\;(2+\delta)\,(3\,\delta+2)=0.$$

Here again we exploit the factored form to split it into the two algebraic sets given respectively by

$$\varepsilon=0,\;(2+\delta)=0$$

and

$$\varepsilon=0,\;3\delta+2=0.$$

Over the first one there are no equilibrium points, and this finishes the computations in this branch. Over the second one we have the representation

$$\begin{array}{l}
-10\,x+9\,x^2+1=0\\
y=\frac{1}{2}(3-3x)\\
z=\frac{1}{2}(x+1)
\end{array}$$

Let us now go back to the equation $\varepsilon\,\delta+\varepsilon+2+\delta=0$. Here again we find another constraint $\varepsilon\;(2+\delta)\neq 0$ so that over the constructible defined by

$$\varepsilon\,\delta+\varepsilon+2+\delta=0,\;\varepsilon\;(2+\delta)\neq 0$$

we have the representation

$$\begin{array}{l}
x=1+\delta\\
y=1\\
z=-\varepsilon v-\delta
\end{array}$$

Over the algebraic set given by

$$\varepsilon\,\delta+\varepsilon+2+\delta=0,\;\varepsilon\;(2+\delta)=0$$

the are no equilibrium points, and this finishes the computations in this branch.

The last equation we have to treat is $\delta = 0$. Here again we need to introduce the constraint $(\varepsilon+2)(\varepsilon+1)\varepsilon \neq 0$ so that over the constructible defined by

$$\delta = 0,\ (\varepsilon+2)(\varepsilon+1)\varepsilon \neq 0$$

the equilibrium points have the representation

$$\begin{aligned} & y^3 - y^2\varepsilon - 3\,y^2 + y\varepsilon + 2\,y = 0 \\ & x = 1 \\ & z = -\varepsilon^2 v y^2 - 3\,\varepsilon\, v y^2 - 2\,v y^2 + 2\,\varepsilon^2 v y + 6\,v y \varepsilon + 4\,v y \end{aligned}$$

where v stands for the inverse of $(\varepsilon+2)(\varepsilon+1)\varepsilon$. Now it remains to see what happens over the algebraic sets given respectively by

$$\delta = 0,\ \varepsilon = 0,$$

$$\delta = 0,\ \varepsilon + 1 = 0,$$

$$\delta = 0,\ \varepsilon + 2 = 0.$$

For these sets we have the representations

$$\begin{aligned} & y^2 - y = 0 \\ & x = 1 \\ & z = -y \end{aligned}$$

$$\begin{aligned} & x = 1 \\ & y = 0 \\ & z = 0 \end{aligned}$$

$$\begin{aligned} & x = 1 \\ & y = 1 \\ & z = \frac{1}{2} \end{aligned}$$

Now we have a partition of $\mathcal{W}(f)$ into several constructibles. We can for example check whether the given system undergoes a Hopf bifurcation (or any other kind of nonzero eigenvalue bifurcation) by verifying on each constructible. Computations tell us that the given system does not undergo a Hopf bifurcation.

In the present example we remark that the polynomials involved in the rational presentations factor into linear polynomials. This means that we can reduce the questions of bifurcations to linear quantifier elimination which is much easier to achieve than the general case.

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