A Maximum Enhancing Higher-Order Tensor Glyph

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Abstract

Glyphs are a fundamental tool in tensor visualization, since they provide an intuitive geometric representation of the full tensor information. The Higher-Order Maximum Enhancing (HOME) glyph, a generalization of the second-order tensor ellipsoid, was recently shown to emphasize the orientational information in the tensor through a pointed shape around maxima. This paper states and formally proves several important properties of this novel glyph, presents its first three-dimensional implementation, and proposes a new coloring scheme that reflects peak direction and sharpness. Application to data from High Angular Resolution Diffusion Imaging (HARDI) shows that the method allows for interactive data exploration and confirms that the HOME glyph conveys fiber spread and crossings more effectively than the conventional polar plot.

1. Introduction

Due to their multivariate nature, tensors cannot be fully represented as grayscale or color values. Therefore, glyphs are the standard tool to inspect the full information of individual tensors. There exists a considerable choice of second-order tensor glyphs for various applications: Specialized glyphs emphasize different aspects of stress and strain tensors [HYW03]. In Diffusion Tensor Magnetic Resonance Imaging (DT-MRI), ellipsoids are the most traditional type of glyph [BMLB94], but more advanced glyphs based on composite shapes [WMK*99] or superquadrics [Kin04] have been constructed to overcome visual ambiguities that occur when projecting ellipsoids to the two-dimensional image space. Similar glyphs have been used to illustrate the tensor voting process [TTMM04] or selected properties of the Hessian near crease features [KSJESW09], and to visualize the nematic liquid crystal alignment tensor [JKM06].

In contrast, a single glyph dominates the visualization of higher-order tensors: namely, the surface whose distance from the origin in each direction equals the value of the homogeneous form in that same direction (Figure 1 (a)). Despite its widespread use, this glyph lacks an established name. It can be considered a generalization of the Reynolds glyph [HYW03], but that name refers to the particular application to Reynolds stress tensors. In the context of High Angular Resolution Diffusion Imaging (HARDI), it has been referred to as a “parametrized surface” [ÖM03], the “higher-order tensor glyph” [HS05], or the “HARDI glyph” [PPvA*09]. To differentiate it from our novel glyph, we adopt the name “spherical polar plot” [Tuc04].

As shown in Figure 1, the polar plot makes it difficult to examine maxima of the homogeneous form: A perfectly symmetric peak (left column) looks similar to an anisotropic one, which is sharper in some directions than in others, and...
different orientations of anisotropic peaks are difficult to discern (middle and right column). However, such differences play an important role in applications like HARDI, where they indicate anisotropic fiber spread.

To overcome such problems, Schultz et al. [SWS09] recently proposed the Higher-Order Maximum Enhancing (HOME) glyph, which generalizes the second-order tensor ellipsoid. When applied to two-dimensional higher-order tensors that arise in image processing, it has been shown to depict maxima more clearly. In this paper, we turn this definition into a more general visualization tool by clarifying its mathematical properties and the conditions under which it can be used. We also present an optimized implementation that provides an efficient framework for interactive exploration of three-dimensional higher-order tensors (Figure 1 (b)). Finally, we support identification and localization of peaks by a new coloring scheme and present novel application examples on HARDI data.

The remainder of this paper is organized as follows: After reviewing related work in Section 2, we introduce the Higher-Order Maximum Enhancing glyph and discuss its formal properties in Section 3. In Sections 4 and 5, we treat coloring by peaks and address implementation issues, respectively. Finally, applications to HARDI data are shown in Section 6, before Section 7 concludes the paper and points out possible directions of future research.

2. Related Work

Several previous authors have tried to visually emphasize maxima in the polar plot: Tuch [Tuc04] visualizes a transformed tensor to emphasize peaks, while trying to avoid exaggerating noise-related features. Hlawitschka and Scheuermann [HS05] render the glyphs semi-transparently and add arrows that point towards the maxima. Similarly, Descoteaux et al. [DDKA09] use lines to indicate maxima. In our work, the shape of the glyph itself is sharpened around the peaks, making them visually more prominent. Perception of the number and direction of maxima is further improved by a new coloring scheme.

Other works have concentrated on interactive rendering of the polar plot: Shattuck et al. [SCB*08] precompute a set of multi-resolution slice images suitable for exploration over the internet. Peeters et al. [PPvA*09] present a GPU-based ray casting approach. We found that, alternatively, a relatively simple geometry-based approach allows for interactive data exploration.

A textbook by Strang [Str98] suggests to visualize a symmetric second-order tensor $T$ by the quadratic surface

$$\{v^T T v = 1\} \quad (1)$$

Corresponding algebraic surfaces have been defined from higher-order tensors by Qi [Q06]. However, since maxima in the homogeneous form lead to minima in these surfaces and vice versa (Figure 1 (c)), they are counter-intuitive in applications like HARDI.

The commonly used tensor ellipsoid, whose half-axes are aligned with the eigenvectors and scaled with the eigenvalues, is obtained by replacing $T$ in Eq. (1) with its squared inverse $T^{-2}$. Özarslan and Mareci [ÖM03] point out that due to the lack of a suitable inverse, this formulation cannot be transferred to higher-order tensors. Therefore, this paper takes a different route to generalizing the tensor ellipsoid, which was first proposed by Schultz et al. [SWS09].

3. Generalizing the Second-Order Tensor Ellipsoid

3.1. Notation

The coefficients $T_{i_1\ldots i_l}$ of an order-$l$ tensor $T$ with respect to a given basis are indexed by $l$ numbers. In this paper, we are concerned with totally symmetric tensors, whose coefficients are invariant under arbitrary index permutations.

The inner product $T \cdot v$ between an order-$l$ tensor $T$ and a vector $v$ produces a tensor $\hat{T}$ of order $(l - 1)$:

$$\hat{T}_{i_1\ldots i_{l-1}} = T_{i_1\ldots i_l} v_i \quad (2)$$

In Eq. (2), the Einstein summation convention implies summation over all values of the repeated index $i_l$. Performing the inner product $l$ times $(\hat{T} \cdot \hat{T} \cdot \ldots \cdot \hat{T} \cdot v)$ produces a scalar. This mapping is called the homogeneous form $F(v)$ of the tensor:

$$F(v) = \hat{T} \cdot v = T_{i_1\ldots i_l} v_i v_{i_2} \ldots v_{i_l} \quad (3)$$

There is a one-to-one mapping between symmetric tensors and their homogeneous forms [CM96], so the full tensor information is conveyed by plotting $F(v)$. Traditionally, this is done using the polar plot

$$p(v) = F(v) v \quad (4)$$

When rewriting $v$ in spherical coordinates, the contribution of the radius $r$ can be separated from the angular part $f(\theta, \phi)$ of the homogeneous form:

$$F(v(r, \theta, \phi)) = f(r, \theta, \phi) = r^l f(\theta, \phi) \quad (5)$$

Therefore, it is sufficient to plot $p(v)$ for vectors on the unit sphere ($\|v\| = 1$). Some authors (including [PPvA*09, DDKA09]) prefer to express $f(\theta, \phi)$ in a spherical harmonics basis of order $l$, with odd-order coefficients set to zero. That basis defines the same space of functions on the sphere as symmetric tensors, and the relation between the respective coefficients is linear [ÖM03]. In this paper, we rely on tensor notation, since it lends itself more easily to a generalization of second-order tensor ellipsoids.

3.2. Generalizing Positive Definiteness

The second-order tensor ellipsoid is only applied to positive definite tensors. A symmetric second-order tensor $T$ is positive definite if its homogeneous form $F(v) > 0$ for all $v \neq 0$.  

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Alternatively, positive definiteness is defined by the fact that all eigenvalues $\lambda_i$ that occur in the spectral decomposition

$$T = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i$$

(6)

are larger than zero. Here, the eigenvectors $e_i$ form an orthonormal basis of the underlying vector space, and both definitions of positive definiteness are equivalent.

Since tensors of odd order exhibit antipodal antisymmetry, $F(-v) = -F(v)$, the notion of positive definiteness is limited to even orders. Similar to symmetric second-order tensors, symmetric higher-order tensors have a decomposition into a sum of rank-1 terms [CGLM08]:

$$T = \sum_{i=1}^{r} \lambda_i e_i \otimes \cdots \otimes e_i$$

(7)

In this case, the vectors $e_i$ can still be normalized to unit length, but they no longer need to be pairwise orthogonal. Consequently, a rank-1 decomposition with all $\lambda_i > 0$ is still a sufficient, but no longer a necessary condition for a positive homogeneous form. Intuitively, negative terms can now be balanced by non-orthogonal positive ones and still create peaks which are so sharp that they cannot be expressed with positive terms alone.

Therefore, in the higher-order case, a positive rank-1 decomposition is a stronger notion of positive definiteness than a positive homogeneous form. For the HOME glyph, we will impose this stronger condition. As illustrated by the self-intersection of the 2D HOME glyph in Figure 2 (b), $F(v) > 0$ is not sufficient to guarantee that the glyph is well-behaved. The reasons for this will become clear in the formal treatment of the glyph properties in Appendix A.

3.3. Definition and Properties of the Glyph

The standard ellipsoid $t(v)$ for symmetric second-order tensors $T$ can be expressed as the transformation of the unit sphere under the linear mapping defined by $T$:

$$t(v) = T \cdot v \quad \text{with} \quad ||v|| = 1$$

(8)

The Higher-Order Maximum Enhancing (HOME) glyph $h(v)$ generalizes Eq. (8) to symmetric tensors of even order $l$. It transforms the unit circle under the mapping given by $(l-1)$ applications of the inner tensor-vector product:

$$h(v) = T \cdot \cdots \cdot v \quad \text{with} \quad ||v|| = 1$$

(9)

The polar plot is related to the HOME glyph through $p(v) = (h(v) \cdot v)v$. Based on two-dimensional experiments in [SWS09], it has been conjectured that at stationary points of the homogeneous form (i.e., points at which the derivative $f'$ vanishes), the following properties hold:

1. The polar plot and the HOME glyph coincide.
2. For both glyphs, the distance to the origin is extremal.
3. The HOME glyph has sharper maxima, while minima are more pronounced in the polar plot.

Intuitively, this is explained by the observation that at non-stationary points, $h(v)$ deflects the input vector $v$ towards a maximum in the homogeneous form. This is also the basis of the power method for finding the largest eigenvector of a matrix, and its generalization to higher-order tensors with a definite homogeneous form [KR02]. When implementing the glyph by deforming a uniformly tessellated sphere, as described in Section 5, it automatically concentrates mesh vertices in the high-curvature regions around sharp peaks. Figure 3 illustrates this in 2D, both for a matrix and a fourth-order tensor.

To formalize these results, and to transfer them to three dimensions, we express $v$ in spherical coordinates to obtain a parametrization of the polar plot $p$ and the HOME glyph $h$ in terms of $\theta$ and $\phi$. Let $h_\theta$ and $h_\phi$ denote partial derivatives of this surface parametrization, $f_\theta$ and $f_\phi$ partial derivatives of the angular part of the homogeneous form. The principal curvatures of $h$ are given as $k_{1h}$ and $k_{2h}$. By the convention in [Top06], maxima have negative principal curvatures.

With this, the three properties of the HOME glyph can be formalized as follows: If $f_\theta(\theta, \phi) = f_\phi(\theta, \phi) = 0$,

1. $p(\theta, \phi) = h(\theta, \phi)$.
2. $\partial |p(\theta, \phi)|/\partial \theta = \partial |p(\theta, \phi)|/\partial \phi = \partial |h(\theta, \phi)|/\partial \theta = \partial |h(\theta, \phi)|/\partial \phi = 0$.
3. $f(\theta, \phi) \neq 0$, $f_\theta(\theta, \phi) \neq 0$, and $f_\phi(\theta, \phi) \neq 0$, then $k_{1h} < k_{2h}$ and $k_{2h} > k_{2h}$.
Establishing these properties ensures that the HOME glyph reproduces extrema of the homogeneous form, and that maxima are indeed sharper than in the polar plot. A formal proof is provided in Appendix A.

4. Coloring by Peaks

Since the full tensor information is expressed in the glyph geometry alone, it is up to the individual application to use color or texture to emphasize specific properties of the tensor field, or to encode additional information.

In diffusion imaging, the degree of anisotropy and the orientation of the tensor are especially relevant, so they are frequently used to determine the color of the glyph. It is customary to employ an XYZ-RGB color coding of the principal direction, in which the spatial x, y, and z axes are mapped to the red, green, and blue color channel, respectively. In this section, we will transfer this way of supporting the perception of tensor orientation with color to higher-order tensors.

4.1. Partitioning of the Glyph Surface

Higher-order tensors may have more than one peak, so it is no longer possible to indicate a single direction by a uniform color. A common trick is to instead color each surface by a measure of linearity in order to avoid indicating a principal direction when it is not clearly defined. One common measure is Westin’s $c_1$ [WMK99], defined from sorted eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$:

$$c_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1}$$

In general, the magnitude of the second derivatives is an indicator of how well-localized a maximum of a function is. For an order-$l$ tensor $T$, the Hessian $H(v)$ of the homogeneous form $F(v)$ at $v$ is given as

$$H(v) = l(l-1)T^{l-2}v$$

Since we are interested in the restriction of $F(v)$ to the unit sphere, we express $v$ in spherical coordinates such that $e_1, e_2$ are orthonormal tangents at the peak. Then, the Hessian $H^{\theta\phi}$ at a stationary point of $f(\theta, \phi)$ is given as

$$H^{\theta\phi} = P^T H P - l f(\theta, \phi) I$$

where $I$ denotes a $2 \times 2$ identity matrix. A derivation of this Equation is given in Appendix B.

Let $\mu_1$ be the eigenvalues of $H^{\theta\phi}$. At a maximum, $\mu_2 \leq \mu_1 \leq 0$. How well a peak is localized depends on the less sharp direction, given by $\mu_1$. As part of Appendix A, it is shown that if the tensor has a positive decomposition, $\mu_1 \geq -l f(\theta, \phi)$ (Eq. (19)). Therefore, $\mu_1$ can be converted into a peak sharpness measure $PS \in [0, 1]$ by normalizing it as follows:

$$PS = \frac{-\mu_1}{l f(\theta, \phi)}$$

Figure 5 presents two examples in which the peak sharpness measure is used to modulate the saturation of XYZ-RGB colors. In (a), the maxima on an almost-planar fourth-order tensor are desaturated strongly. At the same time, the sharp peaks in (b) retain their saturated colors. Despite the fact that saturation varies in the XYZ-RGB colors already,
Figure 5: Color modulation decreases the saturation at broad peaks that lack a clear direction (a). Sharp peaks (b) are still marked by highly saturated colors.

we modulate saturation rather than brightness because dark colors would make it difficult to appreciate glyph shapes.

At the maximum of a second-order tensor, \( f(\theta, \phi) = \lambda_1 \) and \( \mu_1 = 2(\lambda_2 - \lambda_1) \). Thus, peak sharpness coincides with \( c_\ell \) from Eq. (10) for order \( \ell = 2 \). However, \( c_\ell \) characterizes a second-order tensor as a whole, while PS is specific to a particular peak of a higher-order tensor.

Note that we use the name peak sharpness rather than peak anisotropy, since the latter term has been used to denote the degree to which the sharpness of a peak varies as a function of the direction in its tangent plane [KKA07, SCH’07].

5. Interactive Implementation

Our implementation allows for interactive data exploration, even though it is based on standard triangle meshes, along with a few simple optimizations. This may seem surprising in the light of previous work, which used GPU programming for interactive ellipsoid splatting [Gum03] and ray casting of ellipsoids [SWBG06] and polar plots [PPvA’09].

However, unlike in molecule visualization [SWBG06], where the information mainly lies in the spatial relation of the displayed atoms, the utility of tensor glyphs hinges on the discernability of their individual shapes, which requires a larger size in image space. Consequently, the number of geometric objects is generally lower in tensor visualization than in molecule visualization. Moreover, we found the geometry-based implementations used for comparison in part of the literature [PPvA’09] to be highly inefficient.

5.1. Creating the Geometry

Both the polar plot and the HOME glyph can be expressed as deformed spheres, which we approximate by refined icosahedra. In each refinement step, each edge is bisected at its center, thus subdividing each triangle into four smaller ones. New vertices are projected back to the surface of the sphere. To faithfully represent tensors of orders four and six, we use three subdivision steps, generating 642 vertices on the sphere. This corresponds to “tesselation order 7” in [PPvA’09].

Once a suitable triangulation of the sphere is given, glyphs are generated by evaluating Eq.s (4) or (9), respectively. To speed up this process, we exploit the fact that for each vertex \( v \), our tesselation also contains the antipodal point \(-v\). Thus, we only need to compute the transformed positions for half of the vertices and may scale them by \(-1\) to obtain the other half. Moreover, all glyphs are independent from each other, which makes it straightforward to generate them in parallel on modern multi-core CPUs.

Table 1 summarizes the time it takes to generate 5577 glyphs with 642 vertices each from an axial slice of HARDI data on a 2.67 GHz Intel Xeon Quad-Core, for different tensor orders and glyph types. The timings include discrete normal estimation [Tau95]. Even when restricting our implementation to a single CPU core, it creates approximately 27600 glyphs per second. Compared to the 1.15 glyphs per second reported for the same refinement level in [PPvA’09], this amounts to a speedup of factor 24000.

Table 1: Time needed to generate 5577 glyphs, on a quad-core and on a single-core CPU (in brackets).

<table>
<thead>
<tr>
<th>Tensor order</th>
<th>Polar Plot</th>
<th>HOME glyph</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>68 ms (202 ms)</td>
<td>70 ms (224 ms)</td>
</tr>
<tr>
<td>6</td>
<td>76 ms (231 ms)</td>
<td>86 ms (274 ms)</td>
</tr>
</tbody>
</table>

5.2. Coloring

To implement the segmentation-based coloring from Section 4, each local maximum is assigned an XYZ-RGB color, with the saturation modulated according to peak sharpness (Eq. (13)). Then, we move from each uncolored vertex to the neighbor that is furthest from the origin and repeat this recursively until a colored vertex is met. The resulting color is copied to the original vertex and all vertices on the path. This procedure is iterated until no uncolored vertices are left.

In a second pass, the color of each vertex is blended with the colors on its one-ring. For efficiency, we perform simple averaging in RGB space. Coloring the above 5577 glyphs in this way takes 60 ms (210 ms on a single CPU core).

5.3. Efficient Rendering

While exploring a tensor dataset, the user typically either looks at a large-scale overview for orientation, or zooms into a part of the data to examine individual tensors. In the first case, each glyph is projected to a small area on the screen, so coarser tessellations can be used without a visible degradation of image quality. In the second case, only a small number of glyphs is rendered at the same time. Two common acceleration methods exploit this situation: The use of different levels of detail, and culling of objects that fall outside the current view frustum.

We use bounding spheres to implement both techniques. Their radius is given by the global maximum of the homogeneous form, as found during glyph generation. First, glyphs...
outside the viewport are culled by computing signed distances of their center to the six planes that bound the view frustum: If the point is inside and its distance is larger than the radius of the bounding sphere, then the glyph is discarded without further processing.

A high and a low level of detail are given by three and two refinements of the icosahedron, respectively. This way, only one set of vertices has to be kept for each glyph, and the coarser version simply uses a subset of them. The level of detail is chosen based on the projected radius of the bounding sphere. A simple and efficient formula for its approximative computation under perspective projections is derived in Section 3.2 of [LRC+03].

During interactive exploration, the same glyphs are typically rendered many times, from varying positions. In this case, OpenGL Vertex Buffer Objects (VBOs) can be used to keep glyph geometry in video memory. Setting up VBOs for 5577 glyphs takes 55 ms, but speeds up the rendering itself by a factor of \( \approx 2.5 \).

Rendering performance on a 1800 × 1000 viewport using an NVIDIA Quadro FX 1800 is summarized in Table 2. The results correspond to a single slice of HARDI data, and to three orthogonal slices, both at an overview and a zoomed-in level. Even though creating HOME glyphs takes slightly longer than creating polar plots, and order 6 requires more computations than order 4, the rendering of all glyphs is equally efficient. Due to differences in graphics hardware and because the viewport size is not specified, an exact comparison to [PPvA’09] is not possible. However, our results are approximately three orders of magnitude faster than reported for their geometry-based implementation and compare favorably to their ray-tracing based results.

6. Application to HARDI

In High Angular Resolution Diffusion Imaging (HARDI), there exist different types of spherical functions that can be represented as higher-order tensors. The sharpest of them are created by spherical deconvolution [TCGC04] or, equivalently, the “sharpening deconvolution transform” [DDKA09]. In our examples, we focus on orientation density functions (ODFs) that arise from this technique. Therefore, we demonstrate an additional visual sharpening, after a sharpening on the data itself has already been performed.

Unlike diffusion tensors, spherical deconvolution ODFs do not model a physical process. Rather, they describe a fiber distribution which is inferred from the acquired diffusion weighted images (DWIs) based on a particular model of the signal that arises from a single fiber bundle.

6.1. Enforcing a Positive Decomposition

Conceptually, spherical deconvolution ODFs can be expected to meet the positive decomposition assumption from Section 3.2. When using a deconvolution kernel that maps the single fiber response to a rank-1 tensor (as in [SS08]), ODFs that do not possess a positive decomposition correspond to fiber densities that contain negative fiber contributions, which do not make sense anatomically.

In practice, ODFs are not even non-negative, which is commonly attributed to noise. Therefore, we approximate them with a positive sum of rank-1 terms before visualization. This step is analogous to the clamping of negative lobes in the polar plot, which is common in the spherical deconvolution literature [TCGC04].

To perform the positive rank-\(k\) approximation, we make two modifications to the algorithm proposed in [SS08]: First, we replace the heuristic for estimating the number of crossing fibers with a rule that accepts a rank-\(k\) approximation if, compared to the previous rank-(\(k-1\)) approximation, it decreases the norm of the residual by more than \(\varepsilon\) times the original tensor norm. Second, we ensure that only positive rank-1 terms are added.

In most cases, this approximation only leaves a small residual, and the clamped polar plot of the result looks almost identical to the original ODF (cf. Figure 6 (a)). However, using the approximation avoids confusing and misleading self-intersections in the HOME glyph, like they become apparent in 2D in Figure 2(b) and in 3D by the presence of unlit backfaces on the left hand side of Figure 6(b).

Spherical deconvolution ODFs were estimated from 60 DWIs at \(b = 1000\,\text{s/mm}^2\). Figure 7 shows a detail of a coronal slice, at the transition between corpus callosum and pyramidal tract. The top row uses polar plots with clamped negative lobes, the bottom row shows HOME glyphs after positive approximation, with the coloring from Section 4. Two high-resolution images in the supplementary material

Table 2: Framerates at different stages of navigation in a tensor dataset, with and without (in brackets) VBOs.

<table>
<thead>
<tr>
<th># Glyphs</th>
<th># in Level of Detail</th>
<th>Framerate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>5577</td>
<td>5577</td>
<td>0</td>
</tr>
<tr>
<td>5577</td>
<td>0</td>
<td>&gt;100 fps (72 fps)</td>
</tr>
<tr>
<td>12846</td>
<td>12846</td>
<td>0</td>
</tr>
<tr>
<td>12846</td>
<td>299</td>
<td>41 fps (15 fps)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>42 fps (14 fps)</td>
</tr>
</tbody>
</table>

Figure 6: Even when the polar plot hardly shows a difference between the original ODF and a positive rank-\(k\) approximation (a), this step is indispensable for an intersection-free visualization of HOME glyphs (b).
of this paper provide a whole-slice comparison of our novel glyph-based visualization against the state of the art.

In order to determine which glyph conveys the properties of fiber distributions more effectively, we will first consider a number of synthetic examples, for which the underlying fiber configurations are known, before we return to real data.

6.2. Inspecting Fiber Spread

In addition to inferring dominant fiber directions, spherical deconvolution enables conclusions about fiber spread, i.e., the variance around the main direction. Fiber spread is often stronger in some principal direction than orthogonal to it, and it has been proposed that exploiting such information can reduce the leakage of probabilistic tractography methods into anatomically implausible regions [SCH+07].

Anisotropic fiber spread has been modeled using the Bingham distribution [KKA07, SCH+07]:

$$p(v, B) = \frac{1}{C(B)} e^{-\lambda_2 B v}$$  \hspace{1cm} (14)

where $B$ is a symmetric $3 \times 3$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and corresponding eigenvectors $e_1, e_2, e_3$, and $C(B)$ is a constant that makes $p(v, B)$ integrate to unity over the unit sphere. We set $\lambda_1 = 0$, since $p(v, B)$ is invariant under addition of any real constant to the eigenvalues of $B$. Then, $e_1$ is the dominant fiber direction, $e_2$ is the principal direction of fiber spread, and the magnitudes of $\lambda_2$ and $\lambda_3$ define the amount and anisotropy of fiber spread.

To produce a synthetic model of a spreading fiber, we have drawn 1000 samples on the sphere uniformly at random and generated a single fiber model for each of them, weighted according to the Bingham distribution. From this mixture, we then created 60 DWIs at $b = 1000 \text{s/mm}^2$ and estimated a linear spherical deconvolution model of order six.

From left to right, Figure 8 shows fiber spread with $\lambda_2 = \lambda_3 = -15$ (small and isotropic), $\lambda_2 = \lambda_3 = -7.5$ (larger and isotropic), and $\lambda_2 = -5, \lambda_3 = -15$ (anisotropic), respectively. Polar plots (top) and HOME glyphs (bottom) are shown both from the side and by looking orthogonally onto the peak, to maximize the impression of peak anisotropy.

In the polar plot, the width of the lobe grows with the magnitude of its peak. Since less fiber spread leads to a larger density at the dominant direction, the lobe becomes wider as fiber spread decreases, which is counter-intuitive. This problem does not occur in the HOME glyph, where higher variance around the main direction leads to a wider peak. Moreover, the polar plot makes it very difficult to appreciate the spread anisotropy, whereas the aspect ratio of the lobe of the HOME glyph is close to the ratio of $\lambda_2/\lambda_3$.

6.3. Inspecting Fiber Crossings

Since it is known that ODF maxima present a biased estimate of fiber directions in case of crossings [TCG04, SS08], it may appear problematic to use a maximum enhancing glyph for ODFs. To examine this potential problem, we created DWIs of synthetic fiber crossings at varying angles and volume fractions, and again estimated spherical deconvolution models of order six.

Figure 9 compares the resulting ODFs to the fiber directions used in the simulation (gray lines). ODF maxima are highlighted by colored cylinders. It becomes clear that the HOME glyph only develops very sharp, pin-like lobes when maxima are well-separated (left, $78^\circ$ crossing). In such cases, little bias occurs. As the angle becomes smaller, bias increases, but maxima are emphasized less.

The lobe of the HOME glyph on the right ($60^\circ$ crossing) exhibits a steep slope, tangential to the true fiber directions. This is also true for other crossing angles and tensor orders, and is understood by the fact that given a rank-1 decomposition in the form of Eq. (7), the HOME glyph of an order-1...
Figure 9: Results on synthetic crossings show that sharp peaks of the HOME glyph (left) are well-aligned with true fiber directions (gray lines). When maxima are biased (right), glyph shape emphasizes them less.

tensor can be re-written as

$$ h(v) = \sum_{i=1}^{r} \lambda_i (v \cdot e_i)^{l-1} e_i $$  \hspace{1cm} (15)

In case of a two-fiber crossing, $r = 2$, $e_1$ and $e_2$ are the fiber directions (unit-length, but not necessarily orthogonal), and $\lambda_1$ and $\lambda_2$ reflect their respective volume fractions. In Eq. (15), the contribution of $e_1$ is weighted by $\cos(\phi)^{l-1}$, where $\phi$ is the angle between $v$ and $e_1$. Obviously, any vector $v \perp e_1$ is projected unto $e_2$. However, not only is the weight of $e_1$ zero, but also the first $l-2$ derivatives of this weight with respect to $\phi$. Thus, $h(v)$ deviates from the direction of $e_2$ only very slowly as $\phi$ increases. This leads to the good agreement of the glyph shape with the true fiber direction.

On real data, we compared glyph shape to the rank-$k$ approximation from [SS08], which provides unbiased fiber estimates. In this case, we use cylinders to indicate rank-1 terms rather than ODF maxima, and scale their length with relative volume fractions. In these experiments, sharp peaks of the HOME glyph corresponded well to terms of the approximation (Figure 10 (a)).

A priori, it is unclear if wide ODF lobes (Figures 10 (b) and (c)) are due to high fiber spread or to a crossing that has not been resolved by the measurement or the model. Varying the parameters of the approximation allows one to explore both options. We expect that such a combined visualization of ODFs and inferred fiber directions over a region of interest will help to find parameters for model selection that produce coherent and anatomically plausible fiber directions.

Figure 10: In real data (order six), sharp peaks of the HOME glyph (a) correspond well to rank-1 terms. Wide lobes can be interpreted as fiber spread (b) or as unresolved crossings (c).

7. Conclusion and Future Work

The one-to-one relation between ellipsoids and positive definite second-order tensors provides a well-established way of reasoning about these abstract mathematical objects in a graphic and intuitive way. In this work, we have studied the Higher-Order Maximum Enhancing (HOME) glyph, a recent generalization of ellipsoids to higher-order tensors [SWS09]. Our theoretical contributions are a formal proof of its properties and a clarification of the exact conditions under which it can be applied.

Even though the mathematical proofs may appear intimidating, implementing the new glyph is almost as simple as for the polar plot. Contrary to previous findings [PPvA09], we have demonstrated that relatively simple optimizations allow triangle-based implementations to remain interactive up to several ten thousands of glyphs.

Finally, we have made contributions to a particular application, the visualization of data from High Angular Resolution Diffusion Imaging (HARDI). We have transferred the traditional way of coloring diffusion ellipsoids to higher-order tensor models. In a side-by-side comparison to the state of the art, we have demonstrated that the HOME glyph visualizes situations of fiber spread and crossings more effectively than the polar plot that has been used previously.

When discussing our results with a domain scientist, the expert found our approach of assigning colors based on peak directions beneficial and emphasized the importance of interactive glyph-based visualizations for analyzing orientation density functions.

We expect that the HOME glyph will prove helpful both for the exploration of data that can be modelled with higher-order tensors and to clarify more general mathematical properties of higher-order tensors. As part of our future work, we are interested in finding scalar invariants that parameterize the shape of our glyph in an intuitive way.

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Appendix A: Proof of the HOME Glyph Properties

This appendix uses fundamental concepts from differential geometry [Top06] to prove the three properties of the HOME glyph which were stated in Section 3.3. For this, we express $p(v)$ from Eq. (4) and $h(v)$ from Eq. (9) in the parametric form that results from writing $v$ in spherical coordinates:

$$p(\theta, \phi) = f(\theta, \phi) \begin{pmatrix} \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix}$$  \hspace{1cm} (16)

The convention in this paper is to use $\theta \in [0, \pi]$ as the inclination from the positive z axis, and $\phi \in [0, 2\pi]$ as the azimuth angle from the positive x axis.

Standard calculus confirms that

$$T^{-1} v = \frac{1}{T} \nabla F(v)$$  \hspace{1cm} (17)

where $F$ is the homogeneous form of the tensor $T$ (Eq. (3)). We write $F$ in spherical coordinates (cf. $f$ in Eq. (5)) and compute its gradient in Cartesian coordinates via the chain rule. This involves partial derivatives $\partial(r, \theta, \phi)/\partial(x, y, z)$, which can be taken from handbooks like [BSMM04] or from computer algebra software. We observe from Eq. (5) that $\partial f(r, \theta, \phi)/\partial r = r f^{-1}(\theta, \phi)$ and fix $r = 1$ to obtain

$$h(\theta, \phi) = p(\theta, \phi) + \frac{1}{T} \begin{pmatrix} f_0(\theta, \phi) \cos(\theta) \cos(\phi) - f_\phi(\theta, \phi) \sin(\phi) \\ f_0(\theta, \phi) \cos(\theta) \sin(\phi) + f_\phi(\theta, \phi) \cos(\phi) \\ -f_0(\theta, \phi) \sin(\theta) \end{pmatrix}$$  \hspace{1cm} (18)

Without loss of generality, we choose the coordinate system such that the stationary point under investigation is at $(\theta = \pi/2, \phi = 0)$. For brevity, omitting function parameters $\theta$ and $\phi$ will imply function evaluation at that point, and we will omit the terms $f_0 = f_\phi = 0$ completely. Moreover, we rotate the coordinate system such that the mixed second derivatives of $f$ vanish ($f_00 = f_0\phi = 0$).

Under these conditions, $p = h = (f_0, 0, 0)^T$, so the first property follows immediately. For the first derivatives of $p$ and $h$ are $p_0 = (0, 0, -f)^T$, $p_\phi = (0, f, 0)^T$, $h_0 = (0, 0, -f - f_00/2)^T$, and $h_\phi = (0, f + f_00/2, 0)^T$. From this, it is clear that $p \cdot p_0 = p \cdot p_\phi = h \cdot h_0 = h \cdot h_\phi = 0$, which proves property two.

In order to compute principal curvatures, we need to determine the normal vector, $n_0 = h_0 \times h_\phi / \|h_0 \times h_\phi\|$. For this, we will first show that

$$-f - f_00/l \leq 0 \quad \text{and} \quad f + f_0\phi/l \geq 0. \hspace{1cm} (19)$$

The homogeneous form $F_i$ of an individual rank-1 term $T_i = \lambda_i v_i l$ can be written as $F_i(v) = f_i(\xi) = \lambda_i \cos(\xi)$, where $\xi$ denotes the angle between $v$ and $v_i$. Since $f''(\xi) = \lambda_i [-\cos(\xi) + l(l-1) \cos^{l-2}(\xi) \sin^2(\xi)]$, (20) it follows that

$$f''(\xi)/l \geq -f(\xi) \quad \text{if} \quad \lambda_i > 0. \hspace{1cm} (21)$$

According to Section 3.2, we assume that the given tensor can be written as a sum of rank-1 terms with positive $\lambda_i$, so Eqs. (19) hold by linearity of the derivative. Therefore, $n_0 = (1, 0, 0)^T$.

At the stationary point, the coefficients of the first ($E, F, G$) and second fundamental forms ($L, M, N$) are:

$$E_p = f_0^2 \hspace{1cm} E_p = 0 \hspace{1cm} G_p = f_0^2$$
$$L_p = f_00 - f \hspace{1cm} M_p = 0 \hspace{1cm} N_p = f_0\phi - f$$

Since the eigensystems of all fundamental forms coincide, it is simple to compare corresponding principal curvatures. In case of direction $\theta$, the condition for property three reads

$$f_00 - f < \frac{L_0 - 2 f_00}{f_00/l^2}. \hspace{1cm} (22)$$

Straightforward simplification transforms this into $(2l - 1)f + f_00 f_00/l^2 > 0$. Using Eq. (21), it is easy to verify that for $f > 0$ and $f_00 \neq 0$, the condition in Eq. (22) indeed holds. Direction $\phi$ is treated in complete analogy. This concludes the proof of property three.

Appendix B: Hessian in Spherical Coordinates

We obtain second derivatives of the spherical function $f(\theta, \phi)$ from the easy-to-compute Hessian $H$ of the homogeneous function $F(v)$ (Eq. (11)) by expressing $v$ in spherical coordinates, and applying the chain rule to compute

$$\frac{\partial^2 F}{\partial \theta \partial \phi} = \frac{\partial}{\partial \theta} \left( H \frac{\partial F}{\partial \phi} \right) = \nabla F(v) \cdot \frac{\partial^2 v}{\partial \theta \partial \phi}$$  \hspace{1cm} (23)

Taking the second derivative requires both the chain rule and the product rule:

$$\frac{\partial^2 F(v(\theta, \phi))}{\partial \theta \partial \phi} = \frac{\partial^2 v}{\partial \theta \partial \phi} H \frac{\partial v}{\partial \phi} + \nabla F(v) \cdot \frac{\partial^2 v}{\partial \theta \partial \phi}$$  \hspace{1cm} (24)

The other second derivatives are given in complete analogy. Since we chose coordinates such that $v_1 = \partial v/\partial \theta$ and $v_2 = \partial v/\partial \phi$ form an orthonormal basis, $\partial^2 v/\partial \theta \partial \phi = \partial^2 v/\partial \phi \partial \theta = 0$. From property one of the HOME glyph and Equation (17), it is clear that at stationary points of $f(\theta, \phi)$, $\nabla F(v) \cdot v = f(\phi)$. Therefore, collecting all second derivatives in $H^{\theta \phi}$ results in Eq. (12).
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