Distances in the Homology Curve Complex

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Summary

Suppose $S$ is a compact surface with genus at least 2

Question - Given two homologous multicurves $m_1$ and $m_2$ in $S \times \mathbb{R}$, what is the smallest genus, orientable surface with boundary $m_1 - m_2$?

Outline of Talk

- Definition of $\mathcal{HC}(S, \alpha)$
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- The overlap $f$ of two homologous multicurves and algorithms in $\mathcal{HC}(S, \alpha)$
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- Projections to level sets of $f$
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- The overlap $f$ of two homologous multicurves and algorithms in $HC(S, \alpha)$
- Projections to level sets of $f$
- Examples of unusual properties of $HC(S, \alpha)$
“My problems started when I developed a curve complex.”
The Curve Complex

Suppose $S$ is a closed, oriented surface with genus $g \geq 2$.

**Curve Complex $C(S)$ (Harvey)**

The curve complex $C(S)$ is the simplicial complex whose vertex set is the set of all nontrivial free homotopy classes of simple curves on $S$. A collection of vertices $c_1, \ldots, c_k$ span a simplex if and only if $c_1, \ldots, c_k$ can be realised disjointly in $S$.

Distance in $C(S)$ are defined to be the path metric on the 1-skeleton.

It is known that $C(S)$ is
- finite dimensional but **not locally compact**,
- connected (by assumption on $S$), with infinite diameter,
- $\delta$-hyperbolic (Masur-Minsky, Bowditch),
- acted on cocompactly by the mapping class group.
Assorted Complexes and Neuroses

Bestvina-Bux-Margalit - developed the **Complex of Cycles**

Hatcher - the **Cyclic Cycle Complex**

The **Homology Curve Complex**, $\mathcal{HC}(S, \alpha)$, is a simplicial complex whose vertex set is the set of all isotopy classes of oriented multicurves in a given nontrivial element $\alpha$ of $H_1(S, \mathbb{Z})$. A set of vertices $m_1...m_k$ spans a simplex if and only if $m_1, ... m_k$ can be realised disjointly.

Distances in $\mathcal{HC}(S, \alpha)$ are defined to be the path metric on the 1-skeleton.
Figure: Example of a multicurve
Properties of $\mathcal{HC}(S, \alpha)$

**Lemma**

$\mathcal{HC}(S, \alpha)$ is

- Not locally compact
- Contractible (Follows from a construction due to Hatcher)
- Acted on by elements of the mapping class group that preserve $\alpha$.
- Infinite dimensional
- Is not $\delta$-hyperbolic for $g > 3$
- Connected

It will be assumed from now on that $\alpha$ is a primitive homology class.
Intersection number

The intersection number, $i(c, d)$ of two curves is the minimum number of points of intersection between any two elements of the free homotopy classes with representatives $c$ and $d$.

$i_h(c, d)$ is the algebraic intersection number of the homology classes with representatives $c$ and $d$.

If $c$ is an oriented path in $S$ with endpoints $e_1$ and $e_2$ not on $d$, $i_h(c, d)$ is defined to be the algebraic intersection number of $c$ with the homology class in $S \setminus (e_1 \cup e_2)$ with representative $d$. 
Surface Producing Paths

A null homologous multicurve $n$ either “bounds a subsurface of $S$” or “bounds a union of subsurfaces of $S$”.

Figure: A null homologous multicurve that does not bound a subsurface of $S$

In particular, any null homologous multicurve can be decomposed into a union of multicurves that each bound a subsurface of $S$.

Surface producing path

A path in $\mathcal{HC}(S, \alpha)$ passing through the vertices $\gamma_1, \gamma_2, \ldots, \gamma_k$ is surface producing if for each $i$, $\gamma_{i+1} - \gamma_i$ bounds a subsurface of $S$. 
A surface producing path $\gamma_1, \gamma_2, \ldots, \gamma_k$ can be used to construct a cell complex in $S \times \mathbb{R}$ with boundary curves that project onto $\gamma_k - \gamma_1$ by gluing together oriented surfaces with boundary that project 1-1 onto the oriented subsurfaces of $S \cong S \times 0$ bounded by $\gamma_{i+1} - \gamma_i$.

The cell complex so obtained is homotopic to an (immersed) surface in $S \times \mathbb{R}$. 
Definition of $f_{\gamma_k - \gamma_1}$

Figure: $f_{\gamma_k - \gamma_1}(y) - f_{\gamma_k - \gamma_1}(x) := i_h(a, \gamma_k - \gamma_1)$, and set minimum of $f_{\gamma_k - \gamma_1}$ equal to 0.

Well defined, because $i_h(l, \gamma_k - \gamma_1) = 0$ for any closed loop $l$ (algebraic intersection number with anything null homologous is 0).
Figure: A "pre-image" function $g_H$ is obtained that (modulo an additive constant) does not depend on the choice of oriented surface $H$ with boundary $\gamma_k - \gamma_1$
Calculating Distance

$f_{\gamma_k - \gamma_1}$ depends on the choice of representatives $\gamma_1$ and $\gamma_k$ of the free homotopy classes $[\gamma_1]$ and $[\gamma_k]$. Assume $\gamma_1$ and $\gamma_k$ have been chosen such that the maximum of $f_{\gamma_k - \gamma_1}$ is as small as possible.

Counting vertices

The shortest surface producing paths in $HC(S, [\gamma_1])$ connecting $\gamma_1$ and $\gamma_k$ have length $M$, where $M$ is the maximum of $f_{\gamma_k - \gamma_1}$.

Proof: Lower bound - relation between $f_{\partial H}$ and $g_H$

Upper bound - path constructing algorithm.

Hatcher - related path construction algorithm used to prove contractibility of complex of cycles.
Path Construction Algorithm

Given homologous multicurves \( \gamma_1 \) and \( \gamma_k \), use level sets of \( f_{\gamma_k - \gamma_1} \) to construct a surface producing path \( \gamma_1, \gamma_2, \gamma_3, \ldots \gamma_k \) in \( \mathcal{HC}(S, \alpha) \) of length \( M \).

Figure: The subsurface of \( S \) on which \( f_{\gamma_k - \gamma_1} \) is equal to \( M \) defines a surgery that is performed on \( \gamma_1 \) to obtain \( \gamma_2 \).

Similarly, the subsurface of \( S \) on which \( f_{\gamma_k - \gamma_1} \geq M - 1 \) is “the subsurface of \( S \) bounded by \( \gamma_3 - \gamma_2 \)”, etc.
Example
Minimal genus surfaces

**Theorem**

Let $H$ be an orientable surface in $S \times \mathbb{R}$ with smallest possible genus amongst all surfaces with boundary freely homotopic to $\gamma_1 \cup \gamma_k$. $H$ is homotopic to a surface constructed from a s.p. path.

Proof - Morse Theory argument. If $H$ is embedded in $S \times \mathbb{R}$, treat the $R$ coordinate on $H$ as a Morse function. This leads to a handle decomposition of $H$ that corresponds to a path in $HC(S, \alpha)$.

**Corollary**

Suppose $\gamma_1$ and $\gamma_k$ don’t contain null homologous submulticurves or freely homotopic curves, and let $g$ be the smallest possible genus of an orientable surface in $S \times \mathbb{R}$ with boundary $\gamma_k - \gamma_1$. Then there are constants $c_1$ and $c_2$ depending only on $\chi(S)$ such that

$$d_{HC(S)}(\gamma_k, \gamma_1) \leq c_1 g \quad \text{and} \quad g \leq c_2 d_{HC(S)}(\gamma_k, \gamma_1)$$
Subsurface Projections

Masur and Minsky defined “subsurface projection” of \( C(S) \) to a subsurface of \( S \).

Different notion of subsurface projection needed when working in a fixed homology class, because

- Projecting multicurves to arbitrary subsurfaces does not preserve “homologousness”
  (This observation used by Putman to study generators of the Torelli group)
- Distances in \( HC(S, \alpha) \) don’t depend only on distances in subsurface projections, but also on the number of arcs and their relative orientations, i.e.

\[
f_{\lambda \gamma_k - \gamma_1} = \lambda f_{\gamma_k - \gamma_1}
\]

for all integers \( \lambda \). This property extends to distances in subsurface projections of \( HC(S, \alpha) \).
Important Observation

Suppose $\gamma_1$ and $\gamma_k$ are in general position and only have essential points of intersection.

Define $p_1, p_2, \ldots$ as in the diagram. For most $i$, $p_{i+1}$ obtained from $p_i$ by gluing rectangles along the boundary.

Otherwise $i$ is called a critical level of $f_{\gamma_k - \gamma_1}$.
Suppose $\gamma_1, \gamma_2, \gamma_3 \ldots \gamma_k$ is a s.p. path constructed by the algorithm.

Let $l_1 < l_2$ be two integers in the range of $f_{\gamma_k - \gamma_1}$

Sublevel Projection

The sublevel projection of $\gamma_k$ and $\gamma_1$ between the levels $l_1$ and $l_2$ is defined to be

$$\Pi_{l_1}^{l_2}(\gamma_k, \gamma_1) := (\gamma_{l_1}, \gamma_{l_2})$$

Claim - if there are no critical levels between $l_1$ and $l_2$, $\Pi_{l_1}^{l_2}(\gamma_k, \gamma_1)$ is boring.
Examples

Figure: Critical levels 5 and 1

Figure: Critical levels 9, 5 and 1
Lemma

The number of critical levels of \( f_{\gamma_k - \gamma_1} \) is less than \( -\frac{11\chi(S)}{2} - 2 \).

This lemma would follow directly from Gauss-Bonnet, if it weren’t for the fact that the subsurfaces corresponding to the sublevels could be contractible.

Theorem

For all multicurves \( \gamma_1 \) and \( \gamma_k \) that don’t contain freely homotopic curves or null homologous submulticurves, the s.p. paths constructed by the algorithm are (uniform) quasi-geodesics.
Proof Ideas

How many surgeries can be done to $\gamma_i$ to obtain $\gamma_{i+1}$?

Answer - If $\gamma_i$ does not contain freely homotopic curves, a bound depending on $\chi(S)$ is obtained. (Gauss-Bonnet)

Claim - if $\gamma_1$ and $\gamma_k$ don’t contain freely homotopic curves, the algorithm will construct a path $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_k$, where none of the $\gamma_i$ contain freely homotopic curves.

If, in addition there are no critical levels, the algorithm constructs geodesic paths.
The set of all “interesting” sublevel projections
- is uniformly bounded in number
- represent disjoint (possibly contractible) subsurfaces
- inherit a natural ordering

Some applications of sublevel projections
- Locating quasi-flats in $\mathcal{HC}(S, \alpha)$
- Obtaining an algorithm for constructing smallest genus surfaces in $S \times \mathbb{R}$
- Obtaining an algorithm for constructing geodesics in $\mathcal{HC}(S, \alpha)$
- Studying the family of “tight-as-possible” geodesic paths connecting two vertices in $\mathcal{HC}(S, \alpha)$. 
Examples

A family of examples, \((c_n, d_n)\), will be constructed such that:

- \(i(c_n, d_n) = \mathcal{O}(e^n)\)
- \(d_{HC}(S,[c_n])(c_n, d_n) = \mathcal{O}(\sqrt{i(c_n, d_n)})\)
- \(\exists \) only two subsurfaces, \(T_1\) and \(T_2\), to which \(c_n\) and \(d_n\) have large distances in the subsurface projections.

\[
d_{T_1}(c_n, d_n) \text{ and } d_{T_2}(c_n, d_n) < k \log i(c_n, d_n)
\]

- \(c_n := \tau_n(c_1), \ d_n := \omega_n(d_1)\)

Let \(|\tau_n \circ \omega_n^{-1}|\) be the word norm of the element \(\tau_n \circ \omega_n^{-1}\) of the mapping class group in a fixed generating set. Then

\[
\lim_{n \to \infty} \frac{|\tau_n \circ \omega_n^{-1}|}{d_{HC}(S,[c_n])(c_n, d_n)} \to 0
\]
Examples continued

Figure: $c_1$ is homologous to $d_1$, $c_2$ is homologous to $d_2$, and $i(c_1, c_2) = i(d_1, d_2) = 1$

$\psi_a(b) :=$ the curve obtained from $b$ by Dehn twisting around $a$ $k$ times.

$\phi_a(b) :=$ the curve obtained from $b$ by Dehn twisting around $a$ $k + 1$ times.
A family of examples is obtained, where $d_n$ is homologous to $c_n$.

\begin{align*}
c_3 &= \psi_{c_1}(c_2) := \tau_3(c_2) \\
c_4 &= \phi_{c_1}(c_2) := \tau_4(c_2) \\
c_5 &= \psi_{c_3}(c_4) := \tau_5(c_2) \\
c_6 &= \phi_{c_3}(c_4) := \tau_6(c_2) \\
c_7 &= \psi_{c_5}(c_6) := \tau_7(c_2), \text{ etc.}
\end{align*}

\begin{align*}
d_3 &= \psi_{d_1}(d_2) := \omega_3(d_2) \\
d_4 &= \phi_{d_1}(d_2) := \omega_4(d_2) \\
d_5 &= \psi_{d_3}(d_4) := \omega_5(d_2) \\
d_6 &= \phi_{d_3}(d_4) := \omega_4(d_2) \\
d_7 &= \psi_{d_5}(d_6) := \omega_4(d_2), \text{ etc.}
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