Visualizing Tensor Normal Distributions at Multiple Levels of Detail

Amin Abbasloo*, Vitalis Wiens*, Max Hermann, and Thomas Schultz

Abstract—Despite the widely recognized importance of symmetric second order tensor fields in medicine and engineering, the visualization of data uncertainty in tensor fields is still in its infancy. A recently proposed tensorial normal distribution, involving a fourth order covariance tensor, provides a mathematical description of how different aspects of the tensor field, such as trace, anisotropy, or orientation, vary and covary at each point. However, this wealth of information is far too rich for a human analyst to take in at a single glance, and no suitable visualization tools are available. We propose a novel approach that facilitates visual analysis of tensor covariance at multiple levels of detail. We start with a visual abstraction that uses slice views and direct volume rendering to indicate large-scale changes in the covariance structure, and locations with high overall variance. We then provide tools for interactive exploration, making it possible to drill down into different types of variability, such as in shape or orientation. Finally, we allow the analyst to focus on specific locations of the field, and provide tensor glyph animations and overlays that intuitively depict confidence intervals at those points. Our system is demonstrated by investigating the effects of measurement noise on diffusion tensor MRI, and by analyzing two ensembles of stress tensor fields from solid mechanics.

Index Terms—Uncertainty visualization, tensor visualization, direct volume rendering, interaction, glyph based visualization.

1 INTRODUCTION

The visual representation and interactive exploration of errors and uncertainty in three-dimensional scientific visualizations had long been neglected [21], but has recently come into the focus of visualization research [43, 7]. In particular, there has been considerable progress on mathematical modeling and visualization of isosurfaces in uncertain scalar fields [40, 38], and on the extraction of features in uncertain vector fields [34, 37].

In comparison to scalar and vector fields, data uncertainty in tensor fields is more complex and multifaceted, since it can affect different aspects of the tensor field, such as trace, anisotropy, or orientation. Moreover, considering the uncertainty in each of these properties in isolation does not provide a comprehensive picture, since there may be statistical dependencies between them. It is due to this complexity that, even though applications as diverse as mechanical engineering, medical imaging, geometry, computational fluid dynamics, and image processing have benefited from tensor field visualization [10, 31], tools for the visual analysis of uncertainty in those fields are still hard to come by.

Covariance is a fundamental mathematical description of uncertainty in multivariate data, expressing not only the variances in each data channel or coefficient, but also linear dependences between them. If the random variable itself is a second order tensor, covariance is represented by a partially symmetric fourth order tensor [2]. For example, it might indicate variance related to the trace and anisotropy of the tensor, but also the fact that increased trace might be associated with larger anisotropy.

Even in the scalar case, it has been pointed out that visualizations of three dimensional data often have high visual complexity, and care has to be taken to avoid visual clutter when adding uncertainty to them [7]. Clearly, this problem is exacerbated when visualizing uncertain second order tensor fields, which are represented by a second order mean tensor plus a fourth order covariance tensor at each point of three dimensional space. Given this high complexity in data domain and range, it does not seem promising to try and visualize all aspects of tensor covariance simultaneously. Therefore, our work proposes a set of complementary tools that visualize tensor covariance at different
levels of detail, following a strategy that has been phrased by Shneiderman as the visual information seeking mantra “Overview first, zoom and filter, then details on demand” [53].

Our paper is organized as follows: After reviewing related work in Section 2, we introduce the theoretical background of fourth order covariance tensors, which are new to the visualization community, in Section 3. We then present the three pillars of our framework. Section 4 describes overview visualizations, indicating large-scale changes in covariance structures and locations of high variance. Section 5 presents a tool for interactively exploring the exact nature of variability by going back and forth between the spatial domain and the high-dimensional data space created from the second order mean tensor and the fourth order covariance. Finally, Section 6 introduces glyph animations and overlays that, once the analyst decides to focus on a specific point in 3D space, visually represent the full confidence interval there. Throughout the paper, measurement noise in diffusion MRI is used to illustrate our techniques. To emphasize the general applicability of our framework, Section 7 presents an alternative use case, ensemble visualization of stress tensor fields from solid mechanics simulations. Finally, Section 9 concludes the paper.

2 Related Work

Several previous works have visualized specific aspects of uncertainty in tensor fields. Examples are positional uncertainty in isosurfaces of anisotropy measures [41], uncertainty in the direction of the principal eigenvector [22, 49], or in the resulting tensor lines [25, 51, 8, 59, 52].

All these works have focused on visualizing the uncertainty in one specific type of tensor visualization, which is a good strategy to assess to which extent that particular visualization can be relied upon. In contrast, our framework provides a tool that can be used to gain a comprehensive understanding of all aspects of covariance in a given tensor field. For example, it can be used to understand how measurement noise in different acquisition methods affects the whole tensors: Not just specific visualizations, such as tensor lines, but also measures that would be used for quantitative analysis, such as trace or anisotropy.

Even though we are not aware of any previous work that has specifically addressed the visualization of tensor normal distributions, the papers that have introduced them to the medical imaging and signal processing communities have included glyph representations of fourth-order covariance tensors [2, 3]. In Section 6.1, we will discuss them in detail, and demonstrate that our own glyph visualizations provide the additional benefit of illustrating how the covariance affects the mean tensor, effectively visualizing confidence intervals.

Our work specifically addresses data uncertainty in second order tensor fields. Other sources of uncertainty affect tensor visualizations and have been considered in the visualization literature, including uncertainty from parameters in fiber tractography [9] or from choosing between different visualization algorithms [20].

Finally, second order covariance tensors have previously been considered in uncertainty visualization [42, 39]. However, they have been used to encode spatial covariance in scalar fields. In our work, we assume that the random tensors at different points of the field are statistically independent, as it is common when modeling, for example, the effects of thermal measurement noise. Adding spatial correlations would complicate our task significantly, and is left to future work.

3 Theoretical Background

3.1 A Normal Distribution of Tensors

In order to model the uncertainty at each point of the tensor field, our work uses a generalization of the multivariate normal distribution to tensor-valued random variables. It is defined in analogy to the normal distribution of a vectorial random variable $\mathbf{x} \in \mathbb{R}^d$, which, given a mean vector $\bar{\mathbf{x}}$ and a covariance matrix $\Sigma$, can be written as

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})^T \Sigma^{-1} (\mathbf{x} - \bar{\mathbf{x}})\right).$$

A corresponding definition for a symmetric $3 \times 3$ tensorial random variable $\mathbf{D}$, involving a mean tensor $\bar{\mathbf{D}}$ and a fourth order covariance tensor $\Sigma$, has been proposed by Basser and Pajevic [3]:

$$p(\mathbf{D}) = \frac{1}{(2\pi)^{6/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{D} - \bar{\mathbf{D}})^T \Sigma^{-1} (\mathbf{D} - \bar{\mathbf{D}})\right).$$

The “double dot product” operator in Eq. (3) denotes a tensor contraction, so that the expression $\mathbf{D} : \Sigma^{-1} : \mathbf{D} = \Sigma^{-1}_{ijmn} D_{ij} \Sigma_{mn}$ implies summation over all indices of the involved tensors. The fourth order covariance is defined using the tensor product $\otimes$ and the expectation $E$ as

$$\Sigma = E[(\mathbf{D} - \bar{\mathbf{D}}) \otimes (\mathbf{D} - \bar{\mathbf{D}})].$$

The symmetry in the arguments of the tensor product results in the so-called major symmetry $\Sigma_{ijmn} = \Sigma_{mnij}$. Additionally, minor symmetry $\Sigma_{ijmn} = \Sigma_{ijnm}$ follows from the symmetry of the random variable $\mathbf{D}$ itself. Note that this is not the same as full symmetry; for example, $\Sigma_{ijmn} \neq \Sigma_{ijnm}$. Major and minor symmetries reduce the degrees of freedom in $\Sigma$ from $3 \times 3 \times 3 \times 3 = 81$ to just 21.

The benefit of the notation in Eq. (3) is the fact that it fully preserves the tensorial structure of $\mathbf{D}$, while maintaining a close similarity to Eq. (2). However, it involves two non-standard operations: Taking the inverse $\Sigma^{-1}$ of a fourth order tensor, and taking its determinant $|\Sigma|$. These follow naturally from an isometric embedding of $\mathbf{D}$ into $\mathbb{R}^6$. Given tensor coefficients $D_{ij}$, representing $\mathbf{D}$ as a column vector,

$$\mathbf{d} = [D_{11}, D_{22}, D_{33}, \sqrt{2} D_{12}, \sqrt{2} D_{13}, \sqrt{2} D_{23}]^T,$$

preserves the scalar product (for any two tensors $\mathbf{D}^1$ and $\mathbf{D}^2$, $\mathbf{D}^1 : \mathbf{D}^2 = \mathbf{d}^1 : \mathbf{d}^2$) and thus angles and distances.

This vectorization implies a representation of the fourth order covariance tensor $\Sigma$ as a $6 \times 6$ matrix $\mathbf{S}$, given in Table 1. Note that the major symmetry of $\Sigma$ translates to symmetry of the corresponding matrix $\mathbf{S}$. Based on these definitions of $\mathbf{d}$ and $\mathbf{S}$, equivalence of Equations (2) and (3) can be achieved by defining inverse and determinant of $\Sigma$ in terms of the same operations on its matrization $\mathbf{S}$.

Eq. (3) generalizes quite naturally to asymmetric or higher order tensor fields. However, our current framework focuses on the symmetric $3 \times 3$ case, and a full treatment of other cases is left for future work. It is worth noting that, in general, Eq. (3) assigns a non-zero probability to tensors outside the positive definite cone, even if its parameters are estimated from positive definite tensors. In cases where this is undesired, one may easily combine it with the Log-Euclidean

$$\Sigma = \frac{1}{(2\pi)^{6/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{D} - \bar{\mathbf{D}})^T \Sigma^{-1} (\mathbf{D} - \bar{\mathbf{D}})\right).$$

Table 1. This definition represents the $3 \times 3 \times 3 \times 3$ covariance tensor $\Sigma$ as a $6 \times 6$ matrix $\mathbf{S}$. 

$$\mathbf{S} = \begin{pmatrix}
\Sigma_{1111} & \Sigma_{1122} & \Sigma_{1133} & \sqrt{2} \Sigma_{1112} & \sqrt{2} \Sigma_{1113} & \sqrt{2} \Sigma_{1123} \\
\Sigma_{2211} & \Sigma_{2222} & \Sigma_{2233} & \sqrt{2} \Sigma_{2212} & \sqrt{2} \Sigma_{2213} & \sqrt{2} \Sigma_{2223} \\
\Sigma_{3311} & \Sigma_{3322} & \Sigma_{3333} & \sqrt{2} \Sigma_{3312} & \sqrt{2} \Sigma_{3313} & \sqrt{2} \Sigma_{3323} \\
\sqrt{2} \Sigma_{1211} & \sqrt{2} \Sigma_{1222} & \sqrt{2} \Sigma_{1233} & \Sigma_{1212} & \Sigma_{1213} & \Sigma_{1223} \\
\sqrt{2} \Sigma_{1311} & \sqrt{2} \Sigma_{1322} & \sqrt{2} \Sigma_{1333} & \Sigma_{1312} & \Sigma_{1313} & \Sigma_{1323} \\
\sqrt{2} \Sigma_{2311} & \sqrt{2} \Sigma_{2322} & \sqrt{2} \Sigma_{2333} & \Sigma_{2312} & \Sigma_{2313} & \Sigma_{2323}
\end{pmatrix}.$$
A fourth order covariance tensor chosen frame of reference. Unfortunately, it is not easy to interpret than in terms of their individual coefficients, which depend on the terms of their intrinsic properties such as trace or anisotropy, rather when dealing with tensors, it is more intuitive to consider them in a superquadric glyph visualization [48] of the eigentensor effect on different mean tensors. As an example of this, Fig. 2 presents a superquadric glyph visualization [48] of the eigentensor representation of $\Sigma$. Let $B$ be the matrix whose columns result from representing the elements from $\mathcal{B}$ as vectors in $\mathbb{R}^3$ via Eq. (5). Then, the change of basis performed by

$$R = B^T S B$$

results in a rotated representation $R$ of $S$ such that the diagonal elements represent variance related to

1. Changes in trace ($K_1$)
2. Changes in the amount of anisotropy, as measured by $K_2$
3. Changes in the type of anisotropy, as measured by $K_3$
4. – 6. Rotations around the three eigenvectors

The off-diagonal elements of $R$ correspond to the respective covariances. $\mathcal{B}$ is constructed from invariant gradients, i.e., the gradients of $K_3$ with respect to the six-dimensional tensor space, indicating how the tensor coefficients of $\bar{\mathcal{D}}$ have to change in order to change the three invariants $K_1$, most rapidly. $\mathcal{B}$ can be made orthonormal because the gradients of $K_3$ are mutually orthogonal, and can be extended to a basis by adding three rotation tangents, which specify the changes in tensor coefficients caused by infinitesimal rotations of $\bar{\mathcal{D}}$ around its three eigenvectors.

Since $\mathcal{B}$ is constructed from Invariant Gradients and Rotation Tangents, we refer to it as the IGRT frame of reference. Its full derivation and the exact equations are too complex to present here, but can be found in [26, 29]. For the purpose of our work, it is important that $\mathcal{B}$ permits an intuitive interpretation of fourth order covariance tensors in terms of how they affect shape and orientation of the corresponding mean tensor. This is not obvious from the original tensors $\Sigma$, and helps to visualize them in a meaningful way.

4 Overview Visualizations: Slice Views and Direct Volume Rendering

Our framework uses slice views and direct volume rendering to produce overview visualizations that show large-scale spatial structures in the uncertain tensor field and provide a starting point for interactive analysis. The same basic strategy, reducing the high-dimensional data to color, or even a single scalar value, and displaying the result with standard visualization techniques, has been used widely and successfully in tensor field visualization [27, 56, 31]. The challenge in applying this idea to our data is to define suitable summary measures and color maps.

The example dataset on which all our definitions and techniques are illustrated is a brain dataset from diffusion tensor MRI [5] (2 mm isotropic voxel size, 66 uniformly distributed diffusion weighted measurements at \(b = 1000\) s/mm\(^2\) plus one non diffusion weighted reference). Fourth order covariance tensors describing the uncertainty in the tensor estimates under measurement noise have been estimated using the established residual bootstrapping technique [11]. Since the data resolution is relatively coarse (\(112 \times 112 \times 60\) voxels), we upsamplod it with a narrow Gaussian kernel (0.7 voxels standard deviation) to produce a clearer overview of larger-scale structures. The detail visualizations in later sections all use the original data.
4.1 Scalar Summary Measures

A natural question to ask about an uncertain tensor field is in which locations uncertainty is high overall. It can be answered by computing the trace of the covariance matrix from Eq. (1). Based on the spectral decomposition, the trace can be interpreted as the sum of variances along all principal modes of variation. Trace has previously been proposed as a scalar invariant of $\Sigma$ [3], but has not been visualized. Fig. 3 (b) maps its square root and, in (a), compares it to the mean diffusivity of the mean diffusion tensor $D$.

This comparison indicates that overall variance is large in regions of high diffusivity, such as the ventricles (yellow arrows), but also in the thalamus region (red arrows). Anisotropy of the covariance tensor is high in white matter, as shown by the remarkable similarity between our novel Fractional Covariance Anisotropy measure (d) and traditional Fractional Anisotropy of the mean diffusion tensor (c). In contrast, features in FA and FCA differ in a stress tensor field ensemble (e,f).

Fig. 3. In diffusion tensor data, overall variance, as measured by the square root of the covariance trace (b), is large in regions of high mean diffusivity (a), such as the ventricles (yellow arrows), but also in the thalamus region (red arrows). Anisotropy of the covariance tensor is high in white matter, as shown by the remarkable similarity between our novel Fractional Covariance Anisotropy measure (d) and traditional Fractional Anisotropy of the mean diffusion tensor (c). In contrast, features in FA and FCA differ in a stress tensor field ensemble (e,f).

and arbitrary orthonormal tensors $E^2$-$E^6$ that extend $E^1$ to a basis of all symmetric $3 \times 3$ tensors (one such basis is written out in [3]). The intuition behind this fact is that adding or removing the isotropic eigentensor $E^4$ from an isotropic fourth order covariance tensor keeps it isotropic, as does a uniform scaling.

Our novel FCA accounts for this by projecting $\Sigma$ onto the subspace orthogonal to $E^1 \otimes E^1, \Sigma = (E^1 \otimes E^1)/(E^1 \otimes E^1)$, and considering the normalized variance of the remaining five non-trivial eigenvalues of $\Sigma$, which we denote as $\sigma^2_k$, around their mean $\sigma^2$:

$$FCA = \frac{1}{2} \sqrt{\frac{\sum_{k=1}^5 (\sigma^2_k - \sigma^2)^2}{\sum_{k=1}^5 \sigma^2_k}}$$  

This novel measure is visualized in Fig. 3 (d). The fact that it is strikingly similar to standard FA in (c) illustrates that the physics behind diffusion MRI lead to an inevitable link between anisotropic diffusion and anisotropic uncertainty. Since the effective signal to noise ratio depends on the diffusivity, anisotropic diffusion leads to an anisotropic uncertainty in the measurements. Even though this has been known in theory [6], and confirmed using Monte Carlo simulations [3], our novel FCA summary measure allows us, for the first time, to visualize this effect on real-world data, and to visually compare it to the anisotropy of the diffusion tensors themselves.

In tensor fields from other sources, anisotropy of the tensors (FA) and anisotropy of their variance (FCA) may be independent. For example, the green arrow in Fig. 3 (f) highlights a feature in the FCA map that follows the normal distribution in Eq. (3) and a fixed unit vector $x$, the expected value of the Apparent Diffusion Coefficient (ADC) in direction $x$ has been shown to be $x^T D x$ [3], with variance

$$\text{Var}(x^T D x) = (x \otimes x) : \Sigma = (x \otimes x) \sum_{i,j,k,l=1}^3 \sigma_{ijkl} x_i x_j x_k x_l.$$  

Writing out Eq. (13) in index notation makes it obvious that, in this expression, the partially symmetric covariance tensor $\Sigma$ can be replaced with an equivalent totally symmetric tensor $\Sigma$, reducing its 21 independent coefficients to only 15. Since we are not aware of any work that has used this relationship previously, we specify the required component mapping in Appendix A.

The reduced tensor $\Sigma$ contains all information needed to compute variance along any given direction. This totally symmetric part of $\Sigma$
can be visualized with a range of existing techniques for symmetric fourth order tensors \([16, 47, 46]\). In particular, a rank one approximation will readily provide the direction of largest variance \([50]\). It is color coded in Fig. 4 (b), using the XYZ-RGB color map that is the \textit{de facto} standard in diffusion tensor visualization. Comparing it to a standard principal eigenvector map of \(D\) in Fig. 4 (a) shows that the two are in close agreement. Slight variations in intensity result from the fact that colors in (a) have been modulated with FA, while those in (b) have been modulated with FCA, which, as we have seen in the previous subsection, is correlated with, but not identical to FA. This visualization indicates that, in diffusion tensor data, uncertainty in ADC values is usually highest in direction of the fiber (axial diffusivity).

What is lost when reducing \(\Sigma\) to \(\mathcal{S}\) is information about how ADCs in different directions covary. Visualizing this is left to our interactive exploration and glyph based techniques in later sections.

### 4.3 Volume Rendering Tensor Covariance

The novel scalar summary measures and color maps for fourth order covariance tensors that we have derived in the previous subsections follow the widely established ones from diffusion tensor visualization. Beside these, we also provide an overview visualization that takes its inspiration from the concept of an automated default transfer function, which has been proposed by Kniss et al. \([30]\) as a starting point for interactive exploration of scalar volumes.

The goal of default transfer functions is to provide an initial overview of important spatial structures in the data, without requiring the user to design a custom transfer function. For scalar volumes, default transfer functions have been defined by automatically assigning opacity to regions of high gradient magnitude, which often corresponds to boundaries between different materials or objects \([30]\). Gradient magnitude can also be measured in covariance tensor fields: Here, the spatial gradient \(\nabla \Sigma\) is a fifth order tensor, and its magnitude \(\|\nabla \Sigma\|_F\) can simply be computed by taking the Frobenius norm (i.e., the square root of the sum of squares of all its entries).

Since the numerical range of \(\|\nabla \Sigma\|_F\) is not known a priori, we compute a standardized gradient magnitude \(z\) by subtracting the mean over the spatial domain, and dividing by the standard deviation. We then map these normalized values to opacity \(\alpha \in [0, 1]\) using the logistic function

\[
\alpha = \frac{1}{1 + e^{-z-1}}.
\]

The color in default transfer functions has been set by applying a spectral color map to the range of given scalar values \([30]\). For covariance tensors, this does not have an obvious counterpart. While it would be possible to use our color mapping from Section 4.2, or to apply any color map to our scalar measures from Section 4.1, this would imply focusing on one specific aspect of the data, such as overall variance, or direction of largest variance.

Therefore, we opted for a different strategy, which we consider to be more in line with the goal of providing an unbiased, data-driven overview of the most important structures: We use dimensionality reduction to embed the full 21-dimensional fourth order tensors into the perceptually uniform CIELAB color space. Our implementation follows a recent approach by Hamarneh et al. \([15]\). It uses ISOMAP \([55]\), followed by a similarity transformation to fit the resulting 3D embedding into the sRGB gamut.

We expected that our default transfer function would use the full color spectrum to account for the high dimensionality and complexity of the data. Our initial result, presented in Fig. 5 (a), did not meet this expectation: It mainly distinguishes between two dominant regions. Comparing this result to Fig. 3 (b) indicates an agreement with regions of low (greenish) and high (purple) overall variance.

Apparently, these large differences in scale dominate the dimensionality reduction when measuring differences in terms of the Frobenius norm. This effect is removed if we first normalize all covariance tensors by dividing each tensor by its trace. This leads to the more nuanced visualization in Fig. 5 (b), in which major anatomical structures, such as the corpus callosum and cingulum bundle, or the cortical folding become apparent.

Note that interpretation of color in these visualizations differs fundamentally from Fig. 4. Individual colors in Fig. 5 do not stand for any specific property, such as maximum variance in a given direction; rather, colors are chosen such that similar covariance tensors are assigned perceptually similar colors. An example of a conclusion we can draw from the rightmost image of Fig. 5 (b) is that normalized covariance is quite homogeneous over the corpus callosum, which appears in blue, but differs from the one in the cingulum bundle, which is located above it and is more heterogeneous, appearing in cyan, green, reddish, and pink colors.

### 5 Interactive Exploration: How Does Covariance Affect the Mean?

Our summary measures and default transfer functions provide an initial overview, but they consider the covariance tensor \(\Sigma\) in isolation. In order to obtain a full understanding of the uncertain tensor field, we also need to consider the mean tensor \(\bar{D}\) from the tensor normal distribution in Eq. (3). In particular, it is important to keep in mind that, as it was illustrated in Fig. 2, the same covariance \(\Sigma\) can affect the intrinsic properties of different mean tensors \(\bar{D}\) in completely different ways.

Therefore, our tools for interactive exploration make use of the IGRT frame of reference. As explained in Section 3.3, it provides a rotated matrix representation \(\mathbf{R}\) of the covariance that indicates variance associated with changes in trace and anisotropy, and with rotations around the eigenvectors.

The field of symmetric \(6 \times 6\) matrices \(\mathbf{R}\) contains the information we are interested in, but requires us to deal with 21 independent coefficients at each point of three-dimensional space. Our system allows the
5.1 Highlighting Regions of Similar Covariance

The similarity view of our system uses a white-yellow-red color map to indicate how similar the matrix $\mathbf{R}$ at each location $i$ is compared to the matrix $\mathbf{R}_{\text{ref}}$ at a reference voxel, which the user selects by clicking on it. Normalized differences are computed using the Frobenius norm, and mapped to a normalized similarity $s_{\text{norm}} \in [0, 1]$ using the exponential,

$$s_{\text{norm}}(\mathbf{R}_i, \mathbf{R}_{\text{ref}}) = e^{-\frac{\|\mathbf{R}_{\text{ref}} - \mathbf{R}_i\|^2}{\|\mathbf{R}_{\text{ref}}\|^2}}.$$

Our framework also supports an unnormalized similarity,

$$s_{\text{raw}}(\mathbf{R}_i, \mathbf{R}_{\text{ref}}) = e^{-\gamma \|\mathbf{R}_{\text{ref}} - \mathbf{R}_i\|^2},$$

with an additional parameter $\gamma$ that tunes it to the overall scale of $\mathbf{R}$. However, in our uncertain diffusion tensor data, $s_{\text{raw}}$ is dominated by the large differences in covariance trace (shown in Fig. 3 (b)), which makes $s_{\text{norm}}$ more informative.

Our graphical user interface, shown in Fig. 1, supports exploring the three-dimensional structure of similarity maps through three linked orthogonal slice views, and a volume rendering view. Fig. 7 shows three examples of similarity maps, obtained with $s_{\text{norm}}$ and reference voxels within the corticospinal fluid (CSF), gray matter (GM), and white matter (WM). The fact that, in each case, large parts of the same tissue are highlighted as similar indicates that measurement noise in our experimental setup affects the estimated tensors in a manner that is specific to each tissue type. In particular, even though Fig. 3 (d) suggests that covariance is quite isotropic in both gray matter and CSF, Fig. 7 (b/c) reveal that the exact characteristics still differ.

Moreover, comparing Fig. 7 (d) to the standard XYZ-RGB color coding of the mean tensor in Fig. 7 (a) shows that only those parts of the white matter are highlighted as similar that contain a very clear dominant fiber orientation. This makes sense anatomically, since the reference voxel was selected from the corpus callosum. The visualization makes it clear that tensors in the crossing and fanning fibers within the corona radiata are affected differently by the noise. Both observations will be investigated more closely in the next subsection.

5.2 Exploring Individual Aspects of Tensor Covariance

Once the user has discovered spatial regions of similar covariance behavior, it is natural to drill down even deeper and find out the exact extent to which covariance in each of them affects tensor trace, anisotropy, and orientation. To this end, we provide linked views that allow the user to easily switch back and forth between exploring the three-dimensional spatial domain, and the 21-dimensional range of matrices $\mathbf{R}$ at each point. This is similar to the strategy of dual-domain interaction in multi-dimensional transfer function design [30].

The unique coefficients of the normalized symmetric matrix $\frac{\mathbf{R}}{\|\mathbf{R}\|_{F}}$ are visualized in a matrix view, shown in Fig. 6 (a,b). When the user clicks on a voxel, this view is updated to reflect the matrix at the selected location $i$. After normalization, all entries range in $[-1, 1]$, and we use a diverging color map from white to red and blue to indicate positive and negative covariance, respectively. Diagonal elements encode variance, which is always positive. Exact values are displayed as tooltips when the cursor is on an element. As illustrated in Fig. 1, clicking on a matrix element highlights it with a border, and spatially maps that element in the linked slice and volume rendering views, revealing spatial patterns in that specific aspect of tensor covariance.

Fig. 6 presents several findings from using this system. In a mostly isotropic CSF voxel (a), we expected to find the pattern of isotropic covariance discussed in Section 4.1, with one value $\chi_\chi$ for variance aligned with changes in trace (top left element of $\mathbf{R}$), and a second value $\chi_\mu$ that should be the same for all remaining diagonal entries. While this is approximately met by Fig. 6 (a), there is some remaining variance in the second to sixth diagonal elements, and a weak negative covariance between trace and anisotropy. This appears to be the reason why CSF and gray matter, in which the expected isotropic covariance is met almost perfectly, were recognized as distinct groups in Fig. 7.

The fact that normalized variance aligned with trace is found to be relatively small, and nearly uniform over space (c) agrees with results from a previous study, which has found estimates of trace from noisy diffusion MRI to be more reliable than estimates of anisotropy, with only weak dependence on the underlying mean tensor shape [14].

In a white matter voxel from the corpus callosum (b), variance corresponding to changes in the amount of anisotropy is largest, followed by variance corresponding to rotations around the second and third eigenvectors, which together influence the principal eigenvector. We found similar patterns in other regions that are commonly thought to contain a single dominant fiber bundle. The fact that these factors are highly relevant to tractography [33, 4] reinforces the need for high-quality data for reliable fiber tracking [49]. Spatially mapping normalized variance aligned with changes in the amount (d) and type (e) of anisotropy clearly reveals the white matter outlines, confirming that this behavior is present throughout white matter.

Interestingly, we also found notable covariances. In particular, there was a negative relationship between changes in trace and the type of anisotropy. The fact that a previous study has reported a strong negative correlation of trace and anisotropy in patients with ischemic leukoaraiosis [24] makes it particularly noteworthy that a similar correlation, consistent over white matter (f), is already induced by the measurement noise when scanning a healthy volunteer. We also found a correlation between changes in the amount and type of anisotropy.
6.1 Existing Glyphs and Their Limitations

It uses surfaces whose parametric form is given as the radial projection, and we refer to it as a “radial glyph” for brevity.

6.2 Visualizing Confidence Intervals with Animated Tensor Glyphs and Small Multiples

Comparing this definition of \( f(x) \) to Eq. (13) shows that, when applying the radial glyphs to \( \Sigma \), they indicate the variance in each direction. Basser and Pajevic [3] only show this glyph for individual covariance tensors. When using it to visualize uncertainty in tensor fields, we found that the squaring inherent in the definition of \( \Sigma \) leads to large variations in overall glyph size, preventing effective simultaneous visualization of regions of larger and smaller overall variation.

In Fig. 8 (b), we instead apply this glyph to the square root of the covariance \( \sqrt{\Sigma} \), which we define using the spectral decomposition from Section 3.2 by taking \( \sqrt{\Lambda} \) as a diagonal matrix containing the positive square roots of \( \sigma_k^2 \), multiplying out \( \sqrt{S} = EV\sqrt{\Lambda}E^T \) with the eigentensors in \( E \), and mapping \( \sqrt{S} \) back to a fourth order tensor \( \sqrt{\Sigma} \) using Eq. (1). Visualizing \( \sqrt{\Sigma} \) instead of \( \Sigma \) can be considered as a multivariate analogy of showing standard deviation rather than variance.

Comparing the resulting glyphs in Fig. 8 (b) to a superquadric glyph visualization [48] of the mean tensors in (a) shows that, in single fiber regions, variances in (b) mostly appear to be covered by the diffusivity in (a). In Section 4.1, this was explained by the fact that effective signal to noise ratio depends on diffusivity. In regions of partial voluming or fiber crossings, the diffusion tensor model is a poor fit for the measured data, leading to more complex patterns of variance.

In Section 4.2, we have seen that computing variance in all directions only makes use of \( \mathcal{S} \), the totally symmetric part of \( \Sigma \). One consequence of this is that, even though we have seen in Eq. (10) that isotropic covariance tensors have two parameters, they are all visualized as spheres, with only one degree of freedom (radius). In order to distinguish between different types of isotropy, Basser and Pajevic [3] propose an alternative visualization, in which variance plots are applied to each of the six principal modes of variation. To avoid having to inspect six separate images, they superimpose the plots. Fig. 8 (c) presents such a visualization, in which each unit eigentensor has been scaled with the square root of its corresponding eigenvalue.

These composite glyphs suffer from high visual complexity and occlusions. Moreover, they only show the eigentensors, without illustrating how they affect the mean, or indicating confidence intervals around it. As we have seen in Fig. 2, the effect of a perturbation can differ drastically depending on the original tensor.

6.2 Visualizing Confidence Intervals with Animated Tensor Glyphs and Small Multiples

Superquadric glyphs are widely used in diffusion tensor imaging [13], and have also been generalized to indefinite tensors [48]. Confidence intervals of a tensorial probability distribution can be visualized in a natural and intuitive way by showing an animation of such glyphs that, starting at the mean tensor \( \bar{D} \), traverses the smallest part of tensor space that contains the desired fraction of the probability.

It is not obvious what trajectory in the six-dimensional tensor space such an animation should follow. In our framework, we make use of the fact that the spectral decomposition from Section 3.2 identifies six principal modes of variation, which are mutually uncorrelated. Therefore, a good understanding of the six-dimensional confidence interval can be achieved by considering each eigenmode in turn. In particular, the \( k \)th eigenmode is visualized using the family of tensors

\[
D(t; k) = \bar{D} + t\sigma_k E_k^k
\]

where, during animation, \( t \) grows linearly from 0 to \( t_{\text{max}} \), shrinks to \(-t_{\text{max}}\), and returns to 0, before switching to the next eigenmode.
presume, and our visualization framework to investigate where, and in numerical simulation. In this setting, we are using the tensor normal distribution to design a method that can handle an ensemble of indefinite stress tensor fields, which are common in solid mechanics and play an important role in applications such as medical implant planning [12] or mechanical design [32].

According to the well-known properties of the Gaussian distribution, \( t_{\text{max}} = 3 \) corresponds to a 99.7% confidence interval. Example frames from such an animation are shown in Fig. 9.

Even though we found that this animation is quite intuitive, it is sometimes desirable to summarize the confidence interval into a single static image. To this end, it is natural to superimpose glyphs. We found that overlaying only the two most extreme cases along each eigenmode produces sharper results than attempts to achieve a more continuous “motion blur”, such as rasterizing a representative sampling of glyphs, and volume rendering the result [19]. However, naïve superposition of opaque glyphs, as in Fig. 8 (c), leads to occlusion.

We avoid occlusions by rendering the two glyphs separately, in complementary colors (blue and orange), and adding the resulting images. This makes sure that both glyphs are fully visible, a white core is generated in regions where they overlap, and color indicates regions that are only covered by some part of the confidence interval. In Fig. 10, this method is applied to a normally distributed diffusion tensor in a typical single fiber voxel. It shows quite clearly that the largest eigenmode mainly affects axial diffusivity, the second and third rotate the direction of the principal eigenvector, the fourth varies radial diffusivity, and the remaining two have a rather weak effect overall. This finding agrees with the pattern observed in Fig. 6 (b).

However, the small multiples in Fig. 10 clearly depict the exact extent of these changes, which was apparent neither from the matrix views, nor from the eigenmodes in Fig. 8 (c). In the right hand side of Fig. 1, the more intricate pattern of variation that occurs in a crossing fiber voxel is illustrated. In parallel to our work, the Tender glyph has been designed to specifically highlight differences in tensor shape and orientation [60]. It could be used as an alternative to our glyph overlays.

### 7 Application to Stress Tensor Ensembles

In the previous sections, we used data uncertainty in diffusion tensors to illustrate and explain the elements of our framework. In order to demonstrate that our method is applicable to arbitrary fields of normally distributed symmetric \( 3 \times 3 \) tensors, we will now present a second use case, in which the tensor normal distribution is used as a summary statistic for ensembles of indefinite stress tensor fields, which arise in solid mechanics and play an important role in applications such as medical implant planning [12] or mechanical design [32].

#### 7.1 Experimental Setup

We created stress tensor fields by varying different parameters in a numerical simulation. In this setting, we are using the tensor normal distribution from Eq. (3) to summarize the resulting tensor field ensemble, and our visualization framework to investigate where, and in which way, the individual parameters affect the stress tensors.

The simulated system is a quadratic steel plate with side length \( l = 4 \) m and a circular hole with radius \( r = 0.5 \) m at its center. A uniform stretching force is applied along the left and right edges of the plate. Due to symmetry, it is sufficient to simulate the top right quarter of the plate with appropriate boundary conditions, cf. Fig. 11 (a).

According to the well-known properties of the Gaussian distribution, \( t_{\text{max}} = 3 \) corresponds to a 99.7% confidence interval. Example frames from such an animation are shown in Fig. 9.

Even though we found this animation to be very intuitive, it is sometimes desirable to summarize the confidence interval into a single static image. To this end, it is natural to superimpose glyphs. We found that overlaying only the two most extreme cases along each eigenmode produces sharper results than attempts to achieve a more continuous “motion blur”, such as rasterizing a representative sampling of glyphs, and volume rendering the result [19]. However, naïve superposition of opaque glyphs, as in Fig. 8 (c), leads to occlusion.

We avoid occlusions by rendering the two glyphs separately, in complementary colors (blue and orange), and adding the resulting images. This makes sure that both glyphs are fully visible, a white core is generated in regions where they overlap, and color indicates regions that are only covered by some part of the confidence interval. In Fig. 10, this method is applied to a normally distributed diffusion tensor in a typical single fiber voxel. It shows quite clearly that the largest eigenmode mainly affects axial diffusivity, the second and third rotate the direction of the principal eigenvector, the fourth varies radial diffusivity, and the remaining two have a rather weak effect overall. This finding agrees with the pattern observed in Fig. 6 (b).

However, the small multiples in Fig. 10 clearly depict the exact extent of these changes, which was apparent neither from the matrix views, nor from the eigenmodes in Fig. 8 (c). In the right hand side of Fig. 1, the more intricate pattern of variation that occurs in a crossing fiber voxel is illustrated. In parallel to our work, the Tender glyph has been designed to specifically highlight differences in tensor shape and orientation [60]. It could be used as an alternative to our glyph overlays.

#### 7.2 Results from Using our System

In the previous sections, we used data uncertainty in diffusion tensors to illustrate and explain the elements of our framework. In order to demonstrate that our method is applicable to arbitrary fields of normally distributed symmetric \( 3 \times 3 \) tensors, we will now present a second use case, in which the tensor normal distribution is used as a summary statistic for ensembles of indefinite stress tensor fields, which arise in solid mechanics and play an important role in applications such as medical implant planning [12] or mechanical design [32].

We will report results on two experiments: In the first, a Gaussian distributed component tangential to the edge of the plate was added to the external force. It was assumed uniform along the part of the plate included in the simulation, but mirrored along the symmetry axis by the boundary condition. This led to compression at the center of the plate when the tangential component pointed downward (see Fig. 11 (b)), additional stretching when it pointed upward. In the second experiment, the force was kept constant, orthogonal to the edge, but Poisson’s ratio, a material property that reflects how much a material expands in directions perpendicular to a given direction of compression, was varied according to a Gaussian distribution.

Fig. 12 (a) maps \( \sqrt{\text{tr}^2(\Sigma)} \), the square root of covariance trace, when varying the force at the edge of the plate. Numerical values are in the range \([0.49] \text{kPa}\), with the highest value mapped to white. The largest impact is at the center of the edge (bottom right part of the plot due to the symmetry axes), where material is either compressed or stretched, depending on the direction of the parallel component. Interestingly, mapping the directions of maximum variance (b) reveals that it remains aligned with the change in the applied force only in some region along the edge (green), but switches to the other axis of the plate (red) at some point as the force propagates into the material.

In Fig. 12 (c), \( \sqrt{\text{tr}(\Sigma)} \) is shown for the experiment changing Poisson’s ratio. This had a much smaller effect overall, so white in this plot indicates \( \sqrt{\text{tr}(\Sigma)} = 0.67 \) kPa. Here, stresses along the right edge are dominated by the external force, which is constant in this case, and the largest change occurs at the upper boundary of the circular hole.

The direction of main variance is perpendicular to the plate throughout the domain (image not shown), reflecting the fact that stresses in that direction are dominated by the external force, which is constant in this case, and the largest change occurs at the upper boundary of the circular hole.

Using the interaction and glyph views quickly revealed that, in both experiments and in almost all locations, varying the parameters only led to a single significant eigenmode in the variation of stress tensors,
which also explains the high FCA values in Fig. 3 (f). As an example, Fig. 9 shows several frames from an animation that illustrates the change in stress tensors at the center of the right edge of the plate in the first experiment, ranging from mixed compression and tension (left; corresponding to external forces that pull the upper half down, the lower half up) to pure tension (right; corresponding to forces that pull the upper half up, but the lower half down).

8 Discussion and Future Work

Similar to a previous work on tensor glyph design [48], the design of our system is primarily guided by mathematical principles, to ensure that it remains useful across a range of applications. Specifically, we present the first visualization system that accounts for the full covariance information in uncertain symmetric second order tensor fields. We have exemplified its practical utility on two quite different applications, analyzing the uncertainty in diffusion tensor MRI, and analyzing ensembles of stress tensor fields. We expect that our system will prove useful also in other domains in which tensor fields arise, such as geometry processing or fluid flow simulations.

In the specific application context of diffusion MRI, a limitation of dealing with second order tensors lies in their well-known inability to capture the orientations of crossing nerve fiber bundles. Therefore, when answering fundamental scientific questions, high angular resolution, multi-shell, or diffusion spectrum imaging are now preferred [57, 54]. However, the second order DT-MRI model remains popular in clinical practice and in clinical studies [44], due to its much smaller demands on measurement time and hardware. Studying the many factors that affect measurement noise in DT-MRI, including the number and distribution of gradient directions, magnetic field strength, b value, echo time, and use of parallel imaging, remains an active topic in the medical imaging community [23]. In the future, our system could be used to confirm and possibly extend such insights.

A future application that we are even more excited about is the exploratory visualization of DT-MRI ensembles. Whereas traditional DT-MRI group analysis focuses on a few scalar measures such as Fractional Anisotropy or trace, our framework will allow us to account for variations in all aspects of the diffusion tensors. To facilitate such an analysis, it would be natural to extend our current framework in two ways: The first is to account for spatial correlations; the second is to add specific support for the visual comparison of two or more tensor distributions, such as diffusion tensors from different cohorts.

9 Conclusion

The normal distribution is one of the most widely applicable mathematical descriptions of uncertainty. Yet, a framework to describe normal distributions for tensorial random variables, involving a fourth order covariance tensor, has been proposed only recently [2, 3]. Already in the symmetric 3 × 3 case, these distributions have six degrees of freedom in the mean tensor, and another 21 in the fourth order covariance, at each point of three-dimensional space. At the outset of our work, only some basic glyphs, shown in Fig. 8 (b,c), were available to visualize them.

Clearly, the complexity of this data prevents us from fully visualizing it at a single glance. Therefore, our work contributes a range of complementary visualization techniques, permitting a visual analysis that proceeds from coarse to fine levels of detail: Summary measures and default transfer functions provide an overview of large-scale structures; linked views allow for an interactive exploration that identifies regions of common behavior, and to drill down into how the covariance tensor affects intrinsic properties of the mean tensor in each of them. Finally, animated superquadric glyphs and glyph overlays provide an intuition of confidence intervals at individual locations.

We presented two example applications to illustrate the interplay between the different components of our system, and to demonstrate their effectiveness. In particular, exploring data uncertainty caused by measurement noise in DT-MRI has allowed us to gain insights on noise-induced correlations between diffusion tensor measures that, to the best of our knowledge, have not been reported previously.

Acknowledgments

We would like to thank Tobias Schmidt-Wilcke (University Hospital Bergmannsheil, Bochum, Germany) for providing the diffusion MRI dataset. Stress tensors have been simulated with OpenFOAM, www.openfoam.org. Our implementation makes use of Teem, teem.sf.net. Parts of this work have been inspired by discussions at the Dagstuhl seminar 14082, “Visualization and Processing of Higher Order Descriptors for Multi-Valued Data”.

Appendix A: Converting Σ to ρ

The fully symmetric part ρ of the partially symmetric covariance tensor Σ can be obtained by setting \( \rho_{ijkl} = \Sigma_{ijkl} \), with the exception of

\[
\rho_{1122} = \frac{\Sigma_{1122} + \Sigma_{1212}}{3}
\]
\[
\rho_{1133} = \frac{\Sigma_{1133} + \Sigma_{1313}}{3}
\]
\[
\rho_{2233} = \frac{\Sigma_{2233} + \Sigma_{2323}}{3}
\]

References


