Supplementary Document on Solving Trigonometric Moment Problems for Fast Transient Imaging

Christoph Peters[1] Jonathan Klein[2,1] Matthias B. Hullin[1] Reinhard Klein[1]
[1]University of Bonn [2]French-German Research Institute of Saint-Louis

Algorithm 1 Levinson's algorithm.
Input: $\mathbf{b} \in \mathbb{C}^{m+1}$ with B Hermitian and positive-definite.
Output: $\mathbf{c} := B^{-1} \cdot \mathbf{e}_0 \in \mathbb{C}^{m+1}$

- $\mathbf{c}_0 := \frac{1}{b_0}$
- For $l \in \{1, \ldots, m\}$:
  - $d := \sum_{k=0}^{l-1} \mathbf{c}_k \cdot b_{l-k}$
  - $(\mathbf{c}_0, \ldots, \mathbf{c}_l) := \frac{(\mathbf{c}_0, \ldots, \mathbf{c}_{l-1}, 0) - d \cdot (0, \overline{\mathbf{c}_{l-1}}, \ldots, \overline{\mathbf{c}_0})}{1 - |d|^2}$
- Return $(\mathbf{c}_0, \ldots, \mathbf{c}_m)^\mathsf{T}$.

In the following we provide additional computations, algorithms, propositions and proofs to fill in various gaps left in the paper due to space constraints.

8 Signal Formation Model

For description of the signal formation model we have used the following equality:

$$\begin{aligned}
&\frac{1}{T} \cdot \int_0^T (g * s_i)(\tau) \cdot s_s(\tau) \mathrm{d}\tau \\
=&\frac{1}{T} \cdot \int_0^T \int_0^\infty g(t) \cdot s_i(\tau - t) \mathrm{d}t \cdot s_s(\tau) \mathrm{d}\tau \\
=&\int_0^\infty g(t) \cdot \frac{1}{T} \cdot \int_0^T s_i(\tau - t) \cdot s_s(\tau) \mathrm{d}\tau \mathrm{d}t \\
=&\int_0^\infty g(t) \cdot \frac{1}{T} \cdot \int_0^T s_i(\tau) \cdot s_s(\tau + t) \mathrm{d}\tau \mathrm{d}t \\
=&\int_0^\infty g(t) \cdot (s_i \star s_s)(t) \mathrm{d}t
\end{aligned}$$

9 Levinson's Algorithm

The only non-trivial step in evaluation of the maximum entropy spectral estimate is the computation of the vector $B^{-1} \cdot \mathbf{e}_0$. Generic algorithms with run time $O(m^3)$ can be used for this purpose (e.g. a Cholesky decomposition). However, if run time is critical, the special structure of the Toeplitz matrix should be exploited. Levinson's algorithm provides a simple and highly efficient way to do this. It is given in Algorithm 1. Note that $\overline{\mathbf{c}_{l-1}}, \ldots, \overline{\mathbf{c}_0}$ denotes the complex conjugate of $\mathbf{c}_{l-1}, \ldots, \mathbf{c}_0$. A correctness proof can be found in [Burg 1975, p. 14 ff].

Due to its asymptotic run time of $O(m^2)$ Levinson's algorithm is referred to as fast algorithm. Superfast algorithms with an asymptotic run time of $O(m \cdot \log^2 m)$ exist but only outperform Levinson's algorithm for $m \geq 256$ [Ammar and Gragg 1988]. This makes them unattractive in the present scenario where $m < 10$.

10 Achieving Sinusoidal Modulation

In the paper we claim that the modulation used in arccos-phase sampling

$$\frac{1}{n} \cdot \sum_{k=0}^{n-1} s\left(\varphi - \arccos\left(1 - \frac{2 \cdot k + 1}{n}\right)\right) \tag{5}$$

converges to a perfect sinusoidal for $n \to \infty$. We now give the proof. First we observe that Equation (5) is a Riemann-sum and therefore assuming continuous and bounded s

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=0}^{n-1} s\left(\varphi - \arccos\left(1 - \frac{2 \cdot k + 1}{n}\right)\right) \\
=&\int_0^1 s(\varphi - \arccos(1 - 2 \cdot k)) dk.
\end{aligned}$$

Now we apply integration by substitution with $k = \frac{1-\cos(\psi)}{2}$ and obtain

$$\begin{aligned}
&\int_0^\pi s\left(\varphi - \arccos\left(1 - 2 \cdot \frac{1 - \cos(\psi)}{2}\right)\right) \cdot \frac{\sin(\psi)}{2} d\psi \quad (6) \\
=&\frac{1}{2} \cdot \int_0^\pi s(\varphi - \psi) \cdot \sin(\psi) d\psi.
\end{aligned}$$

Thanks to four bucket sampling we can assume that s fulfills $s(\varphi - \pi) = -s(\varphi)$ for all $\varphi \in \mathbb{R}$. Thus, Equation (6) can be rewritten as

$$\frac{1}{4} \cdot \int_0^{2\cdot\pi} s(\varphi - \psi) \cdot \sin(\psi) d\psi = \frac{\pi}{2} \cdot s * \sin(\varphi).$$

By the convolution theorem it follows that all Fourier coefficients except for the ones at index one and minus one vanish, i.e. the arising modulation is perfectly sinusoidal. The sinusoidal component in s is reduced by a factor of $\frac{\pi}{4} \approx 0.79$. This means that harmonic cancellation and arccos-phase sampling have the same asymptotic effect on the demodulation contrast [Payne et al. 2010].

Note that we have not made any statements about the phase shift of the arising modulation. This quantity should be determined through calibration per pixel as described in the paper.

11 Error Bounds

The following result by Karlsson and Georgiou [2013] enables us to compute lower and upper bounds to smoothed densities with prescribed trigonometric moments.

Proposition 2. *Let h be a density such that*

$$\mathbf{b} = \int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \mathrm{d}\varphi.$$

Then for all $\varphi \in [0, 2 \cdot \pi]$

$$h * P_r(\varphi) \geq \frac{1}{2 \cdot \pi} \cdot \left(\Re p(r \cdot \exp(i \cdot \varphi)) - \sqrt{q(r \cdot \exp(i \cdot \varphi))} \right),$$
$$h * P_r(\varphi) \leq \frac{1}{2 \cdot \pi} \cdot \left(\Re p(r \cdot \exp(i \cdot \varphi)) + \sqrt{q(r \cdot \exp(i \cdot \varphi))} \right)$$

with $\Re$ *denoting the real part,*

$$p(z) = \frac{\frac{2}{1-|z|^2} + \mathbf{d}^*(z) \cdot B^{-1} \cdot \mathbf{a}(z)}{\mathbf{a}^*(z) \cdot B^{-1} \cdot \mathbf{a}(z)} \in \mathbb{C},$$
$$q(z) = |p(z)|^2 - \frac{\mathbf{d}^*(z) \cdot B^{-1} \cdot \mathbf{d}(z)}{\mathbf{a}^*(z) \cdot B^{-1} \cdot \mathbf{a}(z)} \in \mathbb{R}$$

and $\mathbf{a}, \mathbf{d} : \mathbb{C} \setminus \{0\} \to \mathbb{C}^{m+1}$ *defined by*

$$\mathbf{a}_j(z) := z^{-j-1}, \quad \mathbf{d}_j(z) := z^{-j-1} \cdot \left(b_0 + 2 \cdot \sum_{k=1}^{j} b_k \cdot z^k \right)$$

for all $j \in \{0, \ldots, m\}$*. These bounds are sharp.*

Proof. By [Karlsson and Georgiou 2013, proof of Proposition 12] we know that there exists a $w_z \in \mathbb{C}$ with

$$\Re w_z = 2 \cdot \pi \cdot h * P_r(\varphi)$$

and

$$|w_z - p(r \cdot \exp(i \cdot \varphi))|^2 \leq q(r \cdot \exp(i \cdot \varphi))$$

and we also know that this bound is sharp. The claim follows immediately. □

12 Estimating Range

In the paper we claim that critical points of the maximum entropy spectral estimate fulfill

$$\sum_{j=0}^{m} \sum_{k=0}^{m} \overline{(B^{-1} \cdot \mathbf{e}_0)_j} \cdot (B^{-1} \cdot \mathbf{e}_0)_k \cdot (j - k) \cdot z^{m+j-k} = 0$$

where $z = \exp(i \cdot \varphi)$. In the following we prove this claim.

For convenience let $\mathbf{c} := B^{-1} \cdot \mathbf{e}_0$. We are looking for critical points of

$$h(\varphi) = \frac{1}{2 \cdot \pi} \cdot \frac{\mathbf{e}_0^\mathsf{T} \cdot B^{-1} \cdot \mathbf{e}_0}{|\mathbf{e}_0^\mathsf{T} \cdot B^{-1} \cdot \mathbf{s}(\varphi)|^2}.$$

These points coincide with the critical points of the denominator

$$|\mathbf{c}^* \cdot \mathbf{s}(\varphi)|^2 = \mathbf{s}^*(\varphi) \cdot \mathbf{c} \cdot \mathbf{c}^* \cdot \mathbf{s}(\varphi) = \sum_{j,k=0}^{m} z^{-k} \cdot \mathbf{c}_k \cdot \overline{\mathbf{c}_j} \cdot z^j.$$

Taking the derivative with respect to z yields

$$\sum_{j,k=0}^{m} \mathbf{c}_k \cdot \overline{\mathbf{c}_j} \cdot (j - k) \cdot z^{j-k-1}.$$

Multiplying by $z^{m+1} \neq 0$ yields the claimed expression.

13 Proof of Theorem 1

The maximum entropy spectral estimate given in Theorem 1 from the paper is at the core of our work. This result has been first proven by John Parker Burg [1975]. In the following we compile a complete proof in a way that we hope will be more accessible to an audience with a background in modern computer graphics. We rely on matrices and algorithmic interpretations wherever possible. Most parts of the proof employ the same ideas as prior works and corresponding references are given within the proofs.

As a preliminary we need one additional definition.

Definition 2. Let $\mathbf{r} : \mathbb{C} \to \mathbb{C}^{m+1}$ with

$$\forall z \in \mathbb{C},\ j \in \{0, \ldots, m\} :\ \mathbf{r}_j(z) = z^j.$$

This map can be understood as generalization of $\mathbf{s}$ to the complex plane because

$$\forall \varphi \in \mathbb{R} :\ \mathbf{s}(\varphi) = \mathbf{r}(\exp(i \cdot \varphi)).$$

For any $\mathbf{c} \in \mathbb{C}^{m+1}$ the expression $\mathbf{c}^* \cdot \mathbf{r}(z)$ is a polynomial of degree m or less. We denote by $\mathbf{e}_j$ the j-th canonical basis vector.

To prove Theorem 1 we also need two lemmata. The first one helps us to prove that the reconstructed density h is well-defined and aids in the computation of its trigonometric moments.

Lemma 1. *Let* $B \in \mathbb{C}^{(m+1)\times(m+1)}$ *be a Hermitian, positive-definite Toeplitz matrix. Then the polynomial* $\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z)$ *has no root* $z_0 \in \mathbb{C}$ *with* $|z_0| \leq 1$*, i.e. all roots lie outside the unit circle.*

Proof. This proof is analogous to [Landau 1987, Proposition 1, p. 51 f.]. Let $z_0 \in \mathbb{C}$ such that $\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z_0) = 0$. Let $\mathbf{c} \in \mathbb{C}^m$ describe the polynomial resulting from division of $\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z)$ by the linear factor $z - z_0$. Then

$$\forall z \in \mathbb{C} :\ \mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z) = (z - z_0) \cdot \begin{pmatrix} \mathbf{c} \\ 0 \end{pmatrix}^* \cdot \mathbf{r}(z)$$
$$\Leftrightarrow \forall z \in \mathbb{C} :\ \mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z) = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}^* \cdot \mathbf{r}(z) - z_0 \cdot \begin{pmatrix} \mathbf{c} \\ 0 \end{pmatrix}^* \cdot \mathbf{r}(z)$$
$$\Leftrightarrow B^{-1} \cdot \mathbf{e}_0 = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix} - \overline{z_0} \cdot \begin{pmatrix} \mathbf{c} \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \overline{z_0} \cdot \begin{pmatrix} \mathbf{c} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix} - B^{-1} \cdot \mathbf{e}_0 \qquad (7)$$

In the following it is useful to consider the dot product and norm induced by the Hermitian, positive definite matrix B:

$$\langle \cdot, \cdot \rangle_B : \mathbb{C}^{m+1} \times \mathbb{C}^{m+1} \to \mathbb{C}$$
$$(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v}^* \cdot B^{-1} \cdot \mathbf{w}$$

$$\|\cdot\|_B : \mathbb{C}^{m+1} \to \mathbb{C}$$
$$\mathbf{v} \mapsto \sqrt{\mathbf{v}^* \cdot B^{-1} \cdot \mathbf{v}}$$

With respect to this dot product $\begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}$ is orthogonal to $B^{-1} \cdot \mathbf{e}_0$:

$$\left\langle \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}, B^{-1} \cdot \mathbf{e}_0 \right\rangle_B = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}^* \cdot B \cdot B^{-1} \cdot \mathbf{e}_0 = \begin{pmatrix} 0 \\ \mathbf{c} \end{pmatrix}^* \cdot \mathbf{e}_0 = 0 \quad (8)$$

Algorithm 2 Inverse of Levinson's algorithm.
Input: $\mathbf{c} = B^{-1} \cdot \mathbf{e}_0 \in \mathbb{C}^{m+1}$ where B is a Hermitian, positive-definite Toeplitz matrix.
Output: $\mathbf{b} := B \cdot \mathbf{e}_0$.

1. $L := 0 \in \mathbb{C}^{(m+1)\times(m+1)}$
2. For $l \in \{m, \ldots, 0\}$:
   (a) $\tilde{\mathbf{c}} := (\overline{\mathbf{c}_l}, \ldots, \overline{\mathbf{c}_0}) \in \mathbb{C}^{l+1}$
   (b) For $j \in \{0, \ldots, l\}$:
      i. $L_{l,j} := \overline{\tilde{\mathbf{c}}_j}$
   (c) $\mathbf{c} := (\mathbf{c}_0, \ldots, \mathbf{c}_{l-1}) - \frac{\mathbf{c}_l}{\tilde{\mathbf{c}}_l} \cdot (\tilde{\mathbf{c}}_0, \ldots, \tilde{\mathbf{c}}_{l-1}) \in \mathbb{C}^l$
3. Compute $\mathbf{b} := L^{-1} \cdot \mathbf{e}_0$ by forward substitution.
4. Return $\mathbf{b}$.

Since B is a Toeplitz matrix, $\begin{pmatrix}0\\ \mathbf{c}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{c}\\ 0\end{pmatrix}$ have the same norm:

$$\left\|\begin{pmatrix}0\\ \mathbf{c}\end{pmatrix}\right\|_B^2 = \sum_{j,k=1}^{m} \overline{\mathbf{c}_{j-1}} \cdot B_{j,k} \cdot \mathbf{c}_{k-1} = \sum_{j,k=0}^{m-1} \overline{\mathbf{c}_j} \cdot B_{j+1,k+1} \cdot \mathbf{c}_k = \sum_{j,k=0}^{m-1} \overline{\mathbf{c}_j} \cdot B_{j,k} \cdot \mathbf{c}_k = \left\|\begin{pmatrix}\mathbf{c}\\ 0\end{pmatrix}\right\|_B^2 \tag{9}$$

We now apply the norm $\|\cdot\|_B^2$ on both sides of Equation (7) to obtain:

$$\left\|\overline{z_0} \cdot \begin{pmatrix}\mathbf{c}\\ 0\end{pmatrix}\right\|_B^2 = \left\|\begin{pmatrix}0\\ \mathbf{c}\end{pmatrix} - B^{-1} \cdot \mathbf{e}_0\right\|_B^2$$
$$\overset{(8)}{\Leftrightarrow} |z_0|^2 \cdot \left\|\begin{pmatrix}\mathbf{c}\\ 0\end{pmatrix}\right\|_B^2 = \left\|\begin{pmatrix}0\\ \mathbf{c}\end{pmatrix}\right\|_B^2 + \left\|B^{-1} \cdot \mathbf{e}_0\right\|_B^2$$
$$\overset{(9)}{\Leftrightarrow} |z_0|^2 = 1 + \frac{\left\|B^{-1} \cdot \mathbf{e}_0\right\|_B^2}{\left\|\begin{pmatrix}\mathbf{c}\\ 0\end{pmatrix}\right\|_B^2} > 1$$

□

The next lemma is concerned with the correctness of Algorithm 2. This algorithm provides the inverse counterpart of Levinson's algorithm (Algorithm 1). It takes the output of the latter algorithm as input and in turn outputs the input of this algorithm. The lemma will be useful for proving that the maximum entropy spectral estimate realizes the given trigonometric moments.

Lemma 2. *Algorithm 2 is correct and terminates in time $O(m^2)$.*

Proof. This proof is analogous to [Landau 1987, Proposition 3, p. 53]. The run time of $O(m^2)$ can be seen directly from the structure of the algorithm. To prove correctness we have to consider main minors of the Toeplitz matrix $B_l := (B_{j,k})_{j,k=0}^{l} \in \mathbb{C}^{(l+1)\times(l+1)}$, i.e. B_l is the top left part of B. We proceed in the following steps:

1. Prove for all $l \in \{0, \ldots, m\}$ that if $B_l \cdot \mathbf{c} = \mathbf{e}_0 \in \mathbb{C}^{l+1}$, then $B_l \cdot \tilde{\mathbf{c}} = \mathbf{e}_l \in \mathbb{C}^{l+1}$,
2. Prove that $B_l \cdot \mathbf{c} = \mathbf{e}_0$ at the beginning of each iteration,
3. Prove $L \cdot B \cdot \mathbf{e}_0 = \mathbf{e}_0$.

Step 1: Suppose that $B_l \cdot \mathbf{c} = \mathbf{e}_0$ and consider the matrix

$$R := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & 1 & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{(l+1)\times(l+1)}.$$

When multiplied from the left, this matrix reverts the order of rows, when multiplied from the right, it reverts the order of columns. In some sense it computes with Hermitian Toeplitz matrices because for all $j, k \in \{0, \ldots, l\}$

$$(B_l \cdot R)_{j,k} = B_{j,m-k} = B_{k,m-j} = \overline{B_{m-j,k}} = \overline{(R \cdot B_l)_{j,k}}.$$

Note that we have exploited $j - (m-k) = k - (m-j)$. It follows that

$$B_l \cdot \tilde{\mathbf{c}} = B_l \cdot R \cdot \overline{\mathbf{c}} = \overline{R \cdot B_l \cdot \mathbf{c}} = \overline{R \cdot \mathbf{e}_0} = \mathbf{e}_l.$$

Step 2: We proceed by induction over l.

Base case, $l = m$: By definition of the input $B_l \cdot \mathbf{c} = \mathbf{e}_0$ holds.

Induction hypothesis: At the beginning of an iteration $B_l \cdot \mathbf{c} = \mathbf{e}_0$ holds.

Inductive step, $l \to l-1$: Consider the state of variables right before execution of Step 2c of Algorithm 2. By the induction hypothesis we know $B_l \cdot \mathbf{c} = \mathbf{e}_0$ and due to Step 1 of the proof also $B_l \cdot \tilde{\mathbf{c}} = \mathbf{e}_l$. The division by $\tilde{\mathbf{c}}_l$ is well-defined because B_l is positive-definite and therefore

$$\tilde{\mathbf{c}}_l = \mathbf{e}_l^* \cdot \tilde{\mathbf{c}} = \mathbf{e}_l^* \cdot B_l^{-1} \cdot \mathbf{e}_l > 0. \tag{10}$$

We observe that the last entry of the vector $\mathbf{c} - \frac{\mathbf{c}_l}{\tilde{\mathbf{c}}_l} \cdot \tilde{\mathbf{c}}$ is zero by construction. It follows that for all $j \in \{0, \ldots, l-1\}$

$$\mathbf{e}_j^* \cdot B_{l-1} \cdot \left((\mathbf{c}_0, \ldots, \mathbf{c}_{l-1}) - \frac{\mathbf{c}_l}{\tilde{\mathbf{c}}_l} \cdot (\tilde{\mathbf{c}}_0, \ldots, \tilde{\mathbf{c}}_{l-1})\right)$$
$$= \mathbf{e}_j^* \cdot B_l \cdot \left(\mathbf{c} - \frac{\mathbf{c}_l}{\tilde{\mathbf{c}}_l} \cdot \tilde{\mathbf{c}}\right) = \mathbf{e}_j^* \cdot \left(\mathbf{e}_0 - \frac{\mathbf{c}_l}{\tilde{\mathbf{c}}_l} \cdot \mathbf{e}_l\right) = \mathbf{e}_j^* \cdot \mathbf{e}_0.$$

In consequence Step 2c of Algorithm 2 updates $\mathbf{c}$ correctly.

Step 3: L is a lower triangular matrix by construction and its diagonal entries are non-zero due to Equation (10). Thus, forward substitution is applicable. For $l \in \{0, \ldots, m\}$ consider the product

$$\mathbf{e}_l^* \cdot L \cdot \mathbf{b} = \begin{pmatrix} B_l^{-1} \cdot \mathbf{e}_l \\ 0 \end{pmatrix}^* \cdot \mathbf{b} = (B_l^{-1} \cdot \mathbf{e}_l)^* \cdot B_l \cdot \mathbf{e}_0 = \mathbf{e}_l^* \cdot \mathbf{e}_0.$$

Thus, $L \cdot \mathbf{b} = \mathbf{e}_0$ and $\mathbf{b} = L^{-1} \cdot \mathbf{e}_0$ as claimed. □

With these preparations at hand we are ready to prove Theorem 1.

Proof for Theorem 1. Since B^{-1} is positive definite, $\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0 > 0$. Thus h takes solely real, positive values. By Lemma 1 Equation (2) is well-defined because

$$\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{s}(\varphi) = \mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(\exp(i \cdot \varphi)) \neq 0.$$

We complete the proof in the following steps:

1. Use Lemma 1 to prove
$$\int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \cdot \mathbf{s}^*(\varphi) \mathrm{d}\varphi \cdot B^{-1} \cdot \mathbf{e}_0 = \mathbf{e}_0, \tag{11}$$
2. Use Lemma 2 to conclude that $\mathbf{b} = \int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \mathrm{d}\varphi$,
3. Prove that the Burg entropy is minimal.

Step 1 (in analogy to [Burg 1975, p. 9 ff.]): We consider entry $j \in \{0, \dots, m\}$ of the vector in Equation (11) individually:

$$
\begin{aligned}
&\mathbf{e}_j^* \cdot \int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \cdot \mathbf{s}^*(\varphi) \mathrm{d}\varphi \cdot B^{-1} \cdot \mathbf{e}_0 \\
=& \int_0^{2\cdot\pi} h(\varphi) \cdot (\mathbf{e}_j^* \cdot \mathbf{s}(\varphi)) \cdot (\mathbf{s}^*(\varphi) \cdot B^{-1} \cdot \mathbf{e}_0) \mathrm{d}\varphi \\
=& \int_0^{2\cdot\pi} \frac{1}{2\cdot\pi} \cdot \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{s}(\varphi)} \cdot (\mathbf{e}_j^* \cdot \mathbf{s}(\varphi)) \mathrm{d}\varphi \\
=& \frac{1}{2\cdot\pi} \cdot \int_{|z|=1} \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z)} \cdot z^j \mathrm{d}z \\
=& \frac{1}{2\cdot\pi\cdot i} \cdot \oint_{|z|=1} \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z)} \cdot z^{j-1} \mathrm{d}z
\end{aligned}
$$

Note that $\oint_{|z|=1}$ denotes a counter-clockwise contour integral over the boundary of the unit circle. Such an integral includes the derivative of the arc length parametrization of the contour as factor. In the present case this is $\frac{\mathrm{d}}{\mathrm{d}\varphi} \exp(i \cdot \varphi) = i \cdot \exp(i \cdot \varphi)$ which is why we divide the integrand by $i \cdot z$.

For $j \geq 1$ the integrand is a holomorphic function within the unit circle because by Lemma 1 the denominator has no roots within the unit circle. By Cauchy's integral theorem such an integral equals zero. For $j = 0$ we employ Cauchy's integral formula:

$$
\begin{aligned}
&\frac{1}{2\cdot\pi\cdot i} \cdot \oint_{|z|=1} \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(z)} \cdot \frac{1}{z} \mathrm{d}z \\
=& \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{r}(0)} = \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0} = 1
\end{aligned}
$$

Thus, Equation (11) does indeed hold.

Step 2: The outer product $\mathbf{s}(\varphi) \cdot \mathbf{s}^*(\varphi)$ yields an $(m+1) \times (m+1)$ matrix where the entry at index $j, k \in \{0, \dots, m\}$ is

$$
\exp(i \cdot (j - k) \cdot \varphi).
$$

It follows that $B' := \int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \cdot \mathbf{s}^*(\varphi) \mathrm{d}\varphi$ is a Hermitian Toeplitz matrix containing the trigonometric moments of h. By Proposition 1 from the paper B' has to be positive definite. Equation (11) implies $B^{-1} \cdot \mathbf{e}_0 = B'^{-1} \cdot \mathbf{e}_0$ and by Lemma 2 this uniquely determines $\mathbf{b} = B \cdot \mathbf{e}_0 = B' \cdot \mathbf{e}_0$. We conclude

$$
\mathbf{b} = \int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \cdot \mathbf{s}^*(\varphi) \cdot \mathbf{e}_0 \mathrm{d}\varphi = \int_0^{2\cdot\pi} h(\varphi) \cdot \mathbf{s}(\varphi) \mathrm{d}\varphi.
$$

Step 3: To prove the optimality of h let $g : [0, 2\cdot\pi] \to \mathbb{R}$ be another non-negative density function with well-defined Burg-entropy $H(g) < \infty$ and $\int_0^{2\cdot\pi} g(\varphi) \cdot \mathbf{s}(\varphi) \mathrm{d}\varphi = \mathbf{b}$. Using Jensen's inequality we obtain:

$$
\begin{aligned}
&\frac{H(h) - H(g)}{2\cdot\pi} \\
=& \frac{1}{2\cdot\pi} \cdot \left(\int_0^{2\cdot\pi} \log g(\varphi) \mathrm{d}\varphi - \int_0^{2\cdot\pi} \log h(\varphi) \mathrm{d}\varphi \right) \\
=& \frac{1}{2\cdot\pi} \cdot \int_0^{2\cdot\pi} \log \frac{g(\varphi)}{h(\varphi)} \mathrm{d}\varphi \\
\leq& \log \frac{1}{2\cdot\pi} \cdot \int_0^{2\cdot\pi} \frac{g(\varphi)}{h(\varphi)} \mathrm{d}\varphi \\
=& \log \int_0^{2\cdot\pi} g(\varphi) \cdot \frac{|\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{s}(\varphi)|^2}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0} \mathrm{d}\varphi \\
=& \log \mathbf{e}_0^* \cdot B^{-1} \cdot \int_0^{2\cdot\pi} g(\varphi) \cdot \frac{\mathbf{s}(\varphi) \cdot \mathbf{s}(\varphi)^*}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0} \mathrm{d}\varphi \cdot B^{-1} \cdot \mathbf{e}_0 \\
=& \log \frac{\mathbf{e}_0^* \cdot B^{-1} \cdot B \cdot B^{-1} \cdot \mathbf{e}_0}{\mathbf{e}_0^* \cdot B^{-1} \cdot \mathbf{e}_0} = \log 1 = 0
\end{aligned}
$$

Thus $H(h) \leq H(g)$ and h is globally minimal. □

References

AMMAR, G. S., AND GRAGG, W. B. 1988. Superfast solution of real positive definite Toeplitz systems. *SIAM Journal on Matrix Analysis and Applications 9*, 1, 61–76.

BURG, J. P. 1975. *Maximum Entropy Spectral Analysis*. Ph.d. dissertation, Stanford University, Department of Geophysics.

KARLSSON, J., AND GEORGIOU, T. 2013. Uncertainty bounds for spectral estimation. *Automatic Control, IEEE Transactions on 58*, 7 (July), 1659–1673.

LANDAU, H. J. 1987. Maximum entropy and the moment problem. *Bull. Amer. Math. Soc. (N.S.) 16*, 1 (01), 47–77.

PAYNE, A. D., DORRINGTON, A. A., CREE, M. J., AND CARNEGIE, D. A. 2010. Improved measurement linearity and precision for AMCW time-of-flight range imaging cameras. *Appl. Opt. 49*, 23 (Aug), 4392–4403.