Technical Section

Embedding shapes with Green’s functions for global shape matching

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We present a novel approach for the calculation of dense correspondences between non-isometric shapes. Our work builds on the well known functional map framework and investigates a novel embedding for the alignment of shapes. We therefore identify points with their Green's functions of the Laplace–Beltrami operator, and hence, embed shapes into their own function space. In our embedding the $L^2$ distances are known as the biharmonic distances, so that our embedding preserves the intrinsic distances on the shape. In the novel embedding each point-to-point map between two shapes becomes and can be represented as an affine map. Functional constraints and novel conformal constraints can be used to guide the matching process.

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1. Introduction

Finding correspondences between two or more shapes is an important sub-task for a variety of applications, in which information has to be transferred or correlated between shapes. For example, local deformations can be transferred for shape editing [1–3], and correlations between corresponding regions can be exploited to compress dynamic meshes [4,5] and to create generative shape models [6–8].

Finding correspondences is especially interesting between non-isometric shapes, see Fig. 1 for some examples. Many previous approaches tailored to register isometric shapes fail in this case. Extrinsic non-rigid ICP [7,9] and variants [10–12] suffer from unreliable correspondences on extrinsic distances and from difficulties in solving the non-linear deformation models. The Blended Intrinsic Maps method [13,14] replaces the extrinsic metric by an intrinsic one and delivers good registration results by assuming the resulting maps to be locally conformal. A problem of BIM is that it is not clear how to incorporate a priori constraints which might be necessary to guide the method to the correct map out of the multiple reasonable ones (Fig. 2). Furthermore, the stitching of local maps leads to inconsistencies at their boundaries. Another group of approaches embeds shapes into a high dimensional space, where $L^2$ distances approximate intrinsic distances. Although most of them allow to incorporate additional constraints, many share the major drawback that their embedding requires a non-linear alignment.

Functional maps [15] overcome this problem by constructing an embedding, in which shapes can be aligned with a linear deformation. Unfortunately $L^2$ distances of delta-distributions, that are typically used to embed points, only approximate intrinsic distances between intrinsically close points, see Fig. 4. This is especially important when only a few implicit constraints are available, such as when matching non-isometric shapes.

In contrast to functional maps [15] we identify points with their Green’s functions of the Laplace–Beltrami operator. In this embedding the $L^2$ distances are the well known biharmonic distances [16], which are an intrinsic distance metric on the shape. They are invariant to isometric shape deformations so that pose deformations have little influence on the matching process. We calculate correspondences by aligning these embeddings with an affine deformation, which can be computed reliably and efficiently. There is a linear relation between the Green’s alignment and the (pull-back) functional map [15], so that we can incorporate functional constraints and operator commutativity into our setting. Last but not least, we can include additional constraints on the alignment, which require the resulting map to be close to conformal.

The main contributions of our paper are (a) a novel embedding of shapes in the functional map framework by identifying points with their Green’s functions, (b) combining our novel embedding with functional constraints and (c) including conformality constraints into functional shape matching.

The paper is organized as follows: Section 2 discussed the related work. Section 3 introduces the alignment of shapes with Green’s functions and relates it to the functional maps framework. We further motivate the novel embedding by a comparing Green’s functions and delta-distributions in Section 4. Then Section 5 shows how to utilize functional constraints and

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Using Green’s works [17,18] we discretize Fig. 1. A variety of results gathered with our method. For each class (fourlegged, teddy, humans, birds) there is a single source (black contour) and a variety of target shapes. Using the sparse correspondences depicted by small spheres we calculate a dense map from the source to each target shape, which we then use to transfer a color field. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

2. Related work

Estimating correspondences between different shapes is a challenging task that has been addressed intensively in literature. In this section, we only provide a brief overview on directly related works and kindly refer the interested reader to the recent surveys [17,18].

2.1. ICP

Initially the problem of shape matching appeared in the context of registering sequential point cloud scans of a static scene. This led to the development of rigid ICP algorithms [19], which alternate between detecting corresponding points and rigidly aligning shapes. Due to the local optimization, these techniques depend strongly on the initial correspondences and on heuristics to prune novel correspondences. A variety of methods extend the original ICP metaphor to match deformed shapes by allowing non-rigid deformations in \( \mathbb{R}^3 \) for the alignment [1,7,9–12,20]. A common shortcoming of these methods is the detection of corresponding points based on extrinsic instead of intrinsic distances. For deformable shapes extrinsic distances can be small even for intrinsically distant points. As a consequence these methods typically require a large number of point-to-point constraints to begin with and utilize sophisticated heuristics to prune novel correspondences.

The Blended Intrinsic Maps method [13] obtains good results by concatenating and blending multiple conformal maps into a single global map. However, it cannot incorporate user constraints, which are sometimes necessary to solve ambiguities (e.g. Fig. 2). Furthermore, at the boundaries of the local maps the results often exhibit discontinuities. Additionally the method is difficult to generalize to point-clouds or shapes of genus other than zero.

Other methods map shapes by parameterizing them on a common domain and then aligning their parameterizations so that either an intrinsic measure of stretch from source to target becomes minimal [21–25] or so that the integrated stretch along a sequence of deformations from the source onto the target [26–28] becomes minimal. These methods deliver continuous maps of high quality, but are often computational demanding and their application on non-simple topologies is non-trivial.

Yet another class of methods uses an ICP-like alignment after embedding shapes into a high dimensional space where \( L_2 \) distances approximate intrinsic ones. Shapes have been embedded with the eigenvectors of an affinity matrix [29,30], with an embedding approximating geodesic distances [31], with an embedding based on electrostatic repulsion [32] and with delta-functions [33]. All of these methods use non-linear maps to align the embeddings. Slightly different are the methods [34,35], where the alignment of shapes is avoided by directly embedding one shape into the other by minimizing a non-linear functional.

2.2. Functional maps

The remarkably successful functional maps framework was introduced in [15]. To the best of our knowledge this paper was the first to fully exploit the fact that a linear alignment of a
functional embedding is sufficient to represent arbitrary non-linear
alignments in $\mathbb{R}^3$. This significantly simplifies the matching
process. Additionally, the authors demonstrate the usefulness of func-
tional constraints, such as matching labeled regions. Particularly
for isometric shapes, where many functional constraints are avail-
able [36,37] and the alignment is a rotation commuting with the
Laplacian, superior results have been achieved. Furthermore the
resulting maps can be optimized with an ICP-like alignment algo-

rithm after embedding the shapes with delta-distributions.

Typically when matching non-isometric shapes there are few
a priori constraints available and the ICP-like alignment becomes
especially important. In this case a drawback of their embedding
emerges, namely the $L_2$ distances on delta-distributions, which are
a critical ingredient for a ICP-like method, are not well-defined.

As we describe in Section 4, the distances are only defined after
projecting the delta-distributions onto the first eigenvectors of the
Laplace operator and the distance metric depends strongly on the
number of eigenvectors that were used. The more eigenvectors are
used the more localized these distances become. In our work we
therefore provide a novel embedding for shape matching that re-
spects intrinsic distances.

Functional maps initiated a series of publications, such as im-
proving the extraction of correspondences [33,38,39,40], utilizing low-
rank assumptions on the functional map [40,41] (which still re-
quires multiple initial constraints typically not available for non-
isometric shapes) or investigating the matching of shape collect-
ions [38,42].

Previous work represented points as Green's functions [16] and
as the related Global Point Signatures [43]. The invariance of the
Dirichlet energy to conformal deformations was prominently ex-
plored in [44]. Yet to the best of our knowledge, we are the first
to compute correspondences by aligning shapes represented with
Green's functions and the first to use functional correspondences
to approximate the functional map representation of a conformal
map.

3. Embedding with Green's functions

We start our exposition by describing the embedding of a
shape $\mathcal{M}$ using Green's functions. Let $L^2(\mathcal{M}) = \{f : \mathcal{M} \rightarrow \mathbb{R} | \int_{\mathcal{M}} f^2(x) \, dx < \infty \}$ be the set of square integrable, real-valued functions
on $\mathcal{M}$ and let $\delta_p$ be the delta-distribution at a point $p$, i.e. $(\delta_p, f) = f(p) \forall f \in L^2(\mathcal{M})$. Let further $\Delta : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ be the Laplace–Beltrami operator and its spectral decomposition have the eigenvalues $\phi_i \in L^2(\mathcal{M})$ and the eigenvalues $\lambda_i \in \mathbb{R}$ ($\lambda_1 \leq \lambda_2 \leq \ldots$). Then the Green's function $g_p$ of the Laplace operator $\Delta$ at point $p$ is the solution of the equation:

$$\Delta g_p = \delta_p$$

This equation has a solution if and only if $\delta_p$ is orthogonal to the null-space of $\Delta$. For a simple exposition, we assume that $0 = \lambda_1 < \lambda_2 \neq 0$, which is the case for a compact, simply-connected shape. Hence the null-space of $\Delta$ is the subspace of constant functions, which is spanned by the first eigenvector $\phi_1$. We further write $\Pi$ for the orthogonal projection on the complement of this null-space, i.e. $\Pi(f) := f - \phi_1 \langle \phi_1, f \rangle$. Therefore in our context we define the Green's function of the Laplace operator of point $p \in \mathcal{M}$ as the solution to the slightly different equation

$$\Delta g_p = \Pi(\delta_p) = \delta_p - \phi_1 \langle \phi_1, \delta_p \rangle \quad \langle g_p, \phi_i \rangle = 0$$

which now always has a solution that can be written with the pseudo-inverse $\Delta^+$ as

$$g_p = \Delta^+ \delta_p = \sum_{i=2}^{\infty} \frac{\phi_i(p)}{\lambda_i} \phi_i.$$  

We state a few properties, for the upcoming discussion:

**Proposition 1.**

(a) The mapping from a surface $\mathcal{M}$ onto its Green's functions $g_p^\mathcal{M}$ is injective, thus an embedding.
(b) The functions $\{ g_p^\mathcal{M}, \phi_1 | p \in \mathcal{M} \}$ form a basis of $L^2(\mathcal{M})$.
(c) The functions $\{ g_p^\mathcal{M} | p \in \mathcal{M} \}$ are linearly dependent and span a subspace of $L^2(\mathcal{M})$ of co-dimension 1 (see the Appendix).

Several Green's functions can be seen in Fig.3. The further points are located on the surface, the more their Green's functions differ. $L_2$ distances on the Green's functions are called biharmonic distances [16] and the left of Fig.3 shows the distance fields of several points.

3.1. Matching shapes

In the following we utilize the properties of Green's functions for the construction of a mapping $T : \mathcal{M} \rightarrow \mathcal{N}$ between the shapes $\mathcal{M}$ and $\mathcal{N}$. As Green's functions define a distance field on the surface, we can recover a point from its Green's function simply by determining the surface point whose Green's function is most similar. Therefore, Green's functions represent the intrinsic location of a point on the shape. We can therefore solve for a map $T$ by solving for a map $G : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$ which aligns the Green's functions of $\mathcal{M}$ onto the Green's functions of $\mathcal{N}$ and only later calculate $T$ from $G$. For clarification we add superscripts to the involved quantities on $\mathcal{M}$ and $\mathcal{N}$. Then such an alignment $G$ has to fulfill:

$$G(g_p^\mathcal{M}) = g_{\Pi(p)}^\mathcal{N}, \forall p \in \mathcal{M}$$

$G$ is well-defined as due to Proposition 1a different points have different Green's functions. Actually there is one unique affine map $G$ which fulfills Eq. (4). It can be written as $G(f) = B(f) + t$ where $B : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{N})$ is a linear map (i.e. $B(f + \lambda g) = B(f) + \lambda B(g)$) $\forall \lambda \in \mathbb{R} \forall f \in L^2(\mathcal{M})$) and $t \in L^2(\mathcal{N})$. Choosing $G$ as an affine map simplifies solving for $G$ and allows the inclusion of least squares constraints. To see why choosing $G$ as an affine map is possible, note that due to Proposition 1b there is a unique linear map $\tilde{G}$ with

$$\tilde{G}(g_p^\mathcal{M} + \phi_1^\mathcal{M}) = g_{\Pi(p)}^\mathcal{N} + \phi_1^\mathcal{N}, \forall p \in \mathcal{M}.$$  

Using $\tilde{G}$ one can write $G(f) = B(f) + t$ as $\tilde{G}(\phi_1^\mathcal{M}) = -\phi_1^\mathcal{N}$ and $B(f) = \tilde{G}(f - (\phi_1^\mathcal{M})(\phi_1^\mathcal{M} + 2t)$ and $G$ is unique, because it can be used to write $\tilde{G}$ using $\tilde{G}(f) = B(f) + (\phi_1^\mathcal{M})(\phi_1^\mathcal{M} + 2t)$ and from $\tilde{G}$ one can infer $T$.

Yet not every map $G$ is induced by some point-wise map $T$. If $G$ is induced by some point-wise map $T$, then it aligns the Green's functions:

$$\forall p \in \mathcal{M}, \exists q_p \in \mathcal{N} : G(g_p^\mathcal{M}) = g_{q_p}^\mathcal{N}$$

On the other hand, if $G$ fulfills the last equation, we can reconstruct $T$ via $T : \mathcal{M} \rightarrow \mathcal{N}, p \mapsto q_p$. The above results are summarized in the following proposition:

**Proposition 2.** For each point-wise map $T$, Eq. (4) defines a unique affine map $G$, that maps the Green's functions of $\mathcal{M}$ onto the Green's functions on $\mathcal{N}$. Such an alignment of Green's functions supports Eq. (6), which can be used to restore $T$ from $G$. Therefore there is an one-to-one relation of point-wise maps $T$ and the Green's alignment $G$ supporting Eq. (6).

3.2. Pullback functional maps

The affine map $G$ is related to the functional maps introduced in [15], which we shortly introduce. Each point-wise map $T : \mathcal{M} \rightarrow \mathcal{N}$ induces a pullback functional map $F$, that pulls function values from $\mathcal{N}$ onto $\mathcal{M}$:

$$F : L^2(\mathcal{N}) \rightarrow L^2(\mathcal{M}) \quad f \mapsto f \circ T$$
Both $F$ as well as $G$ map functions between $\mathcal{M}$ and $\mathcal{N}$ and are thus "functional maps". For the sake of a clear notation we refer to $T$ as a point-wise map, to $F$ as the corresponding pullback functional map and to $G$ as the corresponding alignment of Green's functions. Because $F$ preserves function values at corresponding points, it aligns (dual) delta-distributions ($\cdot^T$ denotes transposed):

$$
(\delta^M_p, F(f)) = (\delta^M_p, f(T(p))) = (\delta^N_{\delta(T)}, f)_N
$$

$$
\Rightarrow \quad F\delta^M_p = \delta^N_{\delta(T)}, \quad \forall p \in \mathcal{M}
$$

\textbf{Note:} The original publication on functional maps [15, Section 6.1] assumes that functional maps defined by Eq. (7) align delta-distributions:

$$
\delta^M_p = F\delta^N_{\delta(T)}, \quad \forall p \in \mathcal{M}
$$

This is true only for area-preserving map, which fulfills $F^{-1} = F^T$. Interestingly the difference of Eqs. (8) and (9) seems to have had limited effect in previous works. One reason might be that for any functional map $F$ defined by Eq. (7) there is another map $F' = F^{-T}$ that aligns delta-distributions. The difference of $F$ and $F'$ depends on the change in the area-form and is often small.

\textbf{Proposition 3.} The alignment of Green's functions $G$ can be written in terms of the pullback functional map $F(A_r, \mathcal{M}$ is the area of $M$, see the Appendix) as

$$
G(f) = \Delta^\frac{1}{M} F^T \Delta^\frac{1}{M} f + \frac{1}{\sqrt{A^M}} \Delta^\frac{1}{M} F^T \phi^M_1
$$

and $F$ can be written in terms of $G(f) = B(f) + t$ as

$$
F = \Delta^\frac{1}{M} B^\frac{1}{N} + \sqrt{A^M} \phi^M_1 T^\frac{1}{N} \Delta^\frac{1}{N} + \frac{A^M}{A^N} \phi^M_1 T N^\frac{1}{N}.
$$

Each constraint on $F$ has an equivalent constraint on $G$ and vice versa. The final correspondences do not depend on whether one optimizes for $F$ or $G$. They depend on the involved constraints to solve for $F$ and $G$ and on the method to extract correspondences from $F$ and $G$.

The point-wise maps $T$ induce only a fraction of all possible functional maps. Not only does any induced functional map align dual delta-distributions (Eq. (8)), but also any functional map $F$ which does so according to Eq. (12) is induced by a point-wise map $T$. Thus, the alignment of dual delta-functions by $F$ and the alignment of Green's functions by $G$ are equivalent conditions. 

$$
\forall p \in \mathcal{M} \exists \delta_p \in \mathcal{N} : F^T \delta^M_p = \delta^N_{\delta_p},
$$

\textbf{Proposition 4.} Let $F$ be a pullback functional map and $G$ an alignment of Green's functions which are related by Eq. (10) or (11), then the following statements are equivalent:

(i) $G$ aligns Green's functions by Eq. (6).

(ii) $F$ aligns dual delta-distributions by Eq. (12).

(iii) There is a point-wise map $T$ inducing $F$ by Eq. (7) or $G$ by Eq. (4).

4. Green's functions vs. delta-distributions

Next we discuss the differences between our novel embedding and the original embedding with delta-distributions and why it matters for non-isometric shape matching. Aligning shapes with an ICP-like algorithm is a minimization of the non-linear ICP energy functional:

$$
E[G, T] = \int_{\mathcal{M}} \|G(p^M_\delta) - g^N_{T(p)}\|^2 dp
$$

This functional depends on the $L_2$ distances in the embedding. For Green's functions these distances are known as biharmonic distances—a well-defined intrinsic distance metric [16]. In contrast, delta-distributions are not square integrable ($\delta^M_\delta \notin L^2(\mathcal{M})$) and their $L_2$ distances are therefore not well-defined. To replace the Green's functions in Eq. (13) with delta-distributions they first have to be projected onto a finite subspace. For example, the authors of [15, Section 6.1] embed the shapes with delta-distributions projected onto the first $k$ eigenvectors of the Laplace operator:

$$
p \in \mathcal{M} \mapsto \delta_{\phi}^k = \sum_{i=1}^k \phi_i(x) \in L^2(\mathcal{M})
$$

Unfortunately, the emerging distances approximate intrinsic distances only for a small number of eigenvectors (e.g. $k < 10$), while for large $k$ the $L_2$ distances do no more approximate intrinsic distances, but only discriminate points:

$$
\left\| g^M_{\phi} - g^N_{\phi} \right\|^2 \leq \sum_{i=1}^k (\phi_i(x) - \phi_i(y))^2 \rightarrow \begin{cases} 0 & x = y \\ \infty & \text{else} \end{cases}
$$

Eventually also the Green's functions have to be approximated by a finite basis. In contrast to $L_2$ distances on delta-distributions the biharmonic distances are well approximated with the first few eigenvectors of the Laplace operator. Approximations of both distance fields with different numbers of eigenvectors are shown in Fig. 4. For 1000 eigenvectors delta-distributions only discriminate the query point in agreement with Eq. (15). The effect of the different embeddings on the alignment process is shown in Fig. 5, where two shapes are matched with an ICP-like method that will be described in Section 7 (no conformality, $\alpha = 0$). Using a small number of eigenvectors ($k = 5$) the results using delta-distributions and Green's functions are similar, while for 20 eigenvectors the alignment with delta-distributions becomes discontinuous, even after iterating. This is especially the case, when only few a priori correspondences are known, which is typical for non-isometric matching.

5. Functional constraints and conformality

Next we utilize the linear relation between $F$ and $G$ to transfer functional constraints and operator commutativity into our setting. Afterwards we propose novel functional constraints for conformal maps.
5.1. Functional constraints

Instead of localizing correspondences at single points, it is often more appropriate to determine corresponding regions, or equivalently to require indicator functions to match. This is an instance of a functional constraint, where the map $T$ is known to pull a function $f_N \in L^2(N)$ back onto another function $h_M \in L^2(M)$:

$$Ff_N = h_M$$

(16)

Functional constraints emerge in other applications as well. For example isometric shapes have the same heat- and wave-kernel-signatures [36,37]. As functional constraints are linear constraints in $F$, they can be written as a linear constraint in $G$ using Eq. (11).

5.2. Operator commutativity

If we have a functional operator on each shape and these operators have the same effect on equivalent functions, then these operators commute with the functional map $F$. In this case applying the first operator followed by a projection onto the other shape has the same effect as first projecting onto the other shape followed by applying the second operator there. A typical example for operator commutativity are intrinsic symmetries and the Laplace-Beltrami operator for isometric shapes where we have:

$$\Delta_M^i F = F \Delta_N^i$$

(17)

Using Eq. (11) this linear constraint on $F$ becomes a linear constraint on $G$.

5.3. Conformal maps

Additionally we introduce novel constraints on $F$ and $G$ by assuming that $T$ approximates a conformal map. A conformal map is a locally angle preserving, continuous map. The class of conformal maps includes the isometric maps and is general enough to match typical shapes of equivalent topology. At the same time a conformal map is already determined by a few known correspondences [14]. Furthermore, conformal maps were successfully used in previous work to match near-isometric shapes [13,14].

Conformality does not only restrict the map $T$, but also the functional map $F$ and the Green’s alignment $G$. For two functions $f, h \in L^2(N)$ the Dirichlet energy is defined by

$$E_D[f,h] = \langle \nabla_N f, \nabla_N h \rangle_N = \langle \Delta_N^i f, h \rangle_N$$

and measures how much their gradients agree. A point-wise map is conformal if and only if its functional representation preserves the Dirichlet energy [44]:

$$\langle \Delta_M^i F(f), F(h) \rangle_M = \langle \Delta_N^i f, h \rangle_N$$

or equivalently

$$F^T \Delta_M = \Delta_N F^{-1}$$

(18)

Eq. (18) is an instance of a non-linear Procrustes problem, namely that $\Delta_M^{1/2} F \Delta_N^{-1/2}$ is orthogonal [45,46]. We propose to use the current functional and point-to-point constraints to transfer Eq. (18) into linear constraints instead.

Each functional constraint $Ff_N = h_M$ is equivalent to $f_N = F^{-1}h_M$, which is combined with Eq. (18) into

$$\langle \Delta_N^i F \Delta_M^i \rangle (f_M) = \Pi_N (f_N)$$

(19)

and each point-to-point constraint $G(g_N^M) = g_M^N$ is equivalent to

$$g_P^* = G^*(g_N^M)$$

which we combined with Eq. (18) into

$$\langle \Delta_M^i F^T \Delta_N^i \rangle (g_N^M (p)) + \frac{1}{\sqrt{A_M}} \Delta_M^i F^{-T} \phi_P^i = g_P^*$$

(20)

where $G^*(f) = \Delta_M^i F^{-T} \Delta_N^i (f) + \frac{1}{\sqrt{A_N}} \Delta_M^i F^{-T} \phi_P^i$ is the inverted Green’s alignment.

Proposition 5. Let $T : M \rightarrow N$ be a conformal, point-wise map with the corresponding functional map $F$ and the corresponding Green’s alignment $G$, then $F$ adheres to Eq. (18). If $F$ fulfills a functional constraint $Ff_N = h_M$, then it also fulfills Eq. (19) and if $G$ fulfills a point-to-point correspondence $G(g_N^M) = g_M^N$, then it also fulfills Eq. (20).

Area preserving maps $(F^T = F^{-1})$ have similar constraints, whose investigation we leave for future work.

5.4. Conformal maps and Green’s functions

It is interesting how Green’s functions change under a conformal map. Let $T : M \rightarrow N$ be a conformal map with the corresponding functional map $F$ and the corresponding Green’s alignment $G$, then the Green’s function of a point $p \in M$ and the Green’s function of the mapped point image $T(p) \in N$ differ by (see the Appendix):

$$F^{-1} g_P^M - g_P^N = \phi_P^N (\phi_N^M, F^{-1} g_P^M) + \Delta_N^i F^i \phi_P^i \frac{1}{\sqrt{A_M}}$$

(21)

$$= \phi_P^N \sum_{i=2}^{\infty} \frac{(F^{-1})_{ii}}{\lambda_i^{M,m}} + \frac{1}{\sqrt{A_M}} \sum_{i=2}^{\infty} \phi_P^i (F)_{ii} \frac{1}{\lambda_i^{N,m}}$$

(22)

The values $F_{ii}$ and $\langle F^{-1} \rangle_{ii}$ represent the mean of the mapped eigenvectors on $N$, which typically are small if the area form changed little, in which case we expect little difference between the original and the mapped Green’s function. Fig. 6 shows the Stanford bunny and multiple increasingly smoothed versions of it. During the smoothing a conformal map between the meshes was obtained [47]. Projecting the Green’s functions from the smoothed meshes onto the original mesh shows that Green’s functions of a single point indeed changed little. Apart from the observation above our work does not exploit the connection of conformal maps and Green’s functions further, but leaves this interesting issue open for future work.

6. Discretization

To represent functions and with them the Green’s alignment $G$ a basis of the function spaces $L^2(M)$ and $L^2(N)$ is required. We use the first $k$ eigenvectors of the Laplace operator $[\phi_1, \phi_2, \ldots, \phi_k]$, which is a common choice in literature [15]. Practical evaluation shows that biharmonic distances can be well approximated with at least six eigenvectors (Fig. 4).

6.1. Spectral decomposition

Next we discuss the calculation of the eigenvalues $\lambda_i$ and eigenvectors $\phi_i$ of a shape’s Laplace-Beltrami operator $\Delta$. Let functions
on the surface be represented by vectors, whose coefficients are the function values at the mesh vertices or point-cloud points. Let W be the matrix representation of the scalar product on functions and L be the matrix representation of the Laplace–Beltrami operator, which is typically written as $L = W^{-1}C$, where C is the Dirichlet energy on functions. The eigenvectors and eigenvalues of L are defined by the following generalized eigenvalue problem:

$$C\phi_i = \lambda_i W\phi_i \text{ so that } \phi_i^T W \phi_j = \delta_{ij} \text{ and } \lambda_1 \leq \lambda_2 \leq \ldots$$

For triangle meshes we utilize the well-known Cotan Laplacian [48,49]. We estimate the vertex areas as a third of the sum of the adjacent triangle areas and set W to be a diagonal matrix of the estimated vertex areas. Furthermore

$$(Cu)_i = \frac{1}{2} \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) (u_i - u_j)$$

(23)

defines the Dirichlet energy, where $\mathcal{N}(i)$ is the 1-neighborhood of the vertex $i$ and $\alpha_{ij}, \beta_{ij}$ are the two angles opposing the edge from vertex $i$ to vertex $j$.

For point-clouds (e.g. Fig. 11a) we use a variant of the Laplace operator from Belkin and Liu [50,51]. They define $L$ so that for a small time $t$ heat-diffusion matches the euclidean one. With the point positions $p_i \in \mathbb{R}^3$ the Dirichlet energy is:

$$(Cu)_i = \frac{1}{4\pi t^2} \sum_j \exp\left(-\frac{||p_i - p_j||^2}{4rt}\right) (u_i - u_j)$$

(24)

$W$ is again the diagonal matrix of the estimated point areas. We estimate the area of a point as $\pi r^2/3$, where $r$ is the average euclidean distance to its six nearest points. This simple heuristic worked well in our experiments and allows us to avoid the more complicated area estimations of Belkin and Liu [50,51].

We choose $t$ so that $\exp(-d^2/4t) = 1/10$, where $d$ is the average euclidean distance of all the points to their 10 closest neighbors. Next we sparsify $C$ by removing small elements as follows. First we mark all coefficients of $C$ larger than $1/(10\cdot4\pi t^2)$. Then in each row we mark the 10 largest coefficients (excluding the diagonal). For symmetry we mark $C_{ij}$ if $C_{ji}$ is marked. Lastly we set all unmarked coefficients to 0 and update the diagonal entries of $C$ so that $C(1, \ldots, 1) = (0, \ldots, 0)^T$.

7. Alignment algorithm

Next we propose a concrete method to calculate a Green’s alignment of two shapes and therefore a point-wise map $T$ from a few known point-to-point or functional constraints.

7.1. Alignment energy

We describe the affine alignment with the variables $B \in \mathbb{R}^{(k-1)\times(k-1)}$; $t, \bar{t} \in \mathbb{R}^{k-1}$ and we define a quadratic energy, where we incorporate point-to-point and functional constraints in a least squares sense. The parameter $\alpha \in [0, 1]$ encodes whether we assume conformity. For the point-to-point correspondences $\mathcal{C} = \{(p_1, q_1), \ldots, (p_n, q_n)\} \in (\mathcal{M} \times \mathcal{N})^n$ the matching energy $E^C_B[B; t, \bar{t}]$ is

$$\frac{1}{n} \sum_{l=1}^n \left( \|B g_{p_l}^M + t - g_{q_l}^N\|_2^2 + \alpha \|\Pi_M F g_{q_l}^N + \bar{t} - g_{p_l}^M\|_2^2 \right)$$

(25)

and for the functional constraints $\mathcal{D} = \{(f_1, h_1), \ldots, (f_m, h_m)\} \in (L^2(\mathcal{M}) \times L^2(\mathcal{N}))^m$ the matching energy $E^\mathcal{D}_B[B; t, \bar{t}]$ is

$$\frac{1}{m} \sum_{l=1}^m \left( \|F h_l^N - f_l^M\|_2^2 + \alpha \|\Delta h_l^N F \Delta f_l^M - \Pi_1 h_l^N\|_2^2 \right)$$

(26)

where $F, B$ and $t$ are related by Eqs. (10) and (11).

7.2. Initial solving

Typically there are so few a priori point-to-point and functional constraints, that in the first iteration the energy is under-constrained. We therefore add further regularization constraints. Translation depends on the area scale and is typically small, so that we assume $t = \bar{t} = 0$. Furthermore the distortion of $G$ should be as little as possible, so that we add the regularizer $\epsilon \|B\|_2^2$ ($\epsilon = 10^{-6}$). In our experiments this simple choice lead to consistently good results and clearly outperformed other possible terms, such as $\|F\|_2^2$ or $\|\Delta^N F F^N\|_2^2$. In conclusion the initial alignment $G$ for the a priori point-to-point $C_0$ and functional constraints $D_0$ is the unique minimum of:

$$E^C_{C_0} [B, 0, 0] + E^\mathcal{D}_{D_0}[B, 0, 0] + \epsilon \|B\|_2^2: \quad t = \bar{t} = 0$$

(27)

7.3. Updating correspondences

Once we have an alignment $G$ we map the Green’s functions of $\mathcal{M}$ onto the Green’s functions of $\mathcal{N}$ and determine novel point-to-point correspondences with a nearest neighbor search:

$$\rho_M: \mathcal{M} \rightarrow \mathcal{N}, \quad p \mapsto \arg \min_{x \in \mathcal{N}} \|B g_p^M + t - g_x^N\|_2$$

$$\rho_N: \mathcal{N} \rightarrow \mathcal{M}, \quad q \mapsto \arg \min_{y \in \mathcal{M}} \|B g_y^N + t - g_q^M\|_2$$

It is sufficient to calculate correspondences from and onto a subset of both shapes $S_M$ and $S_N$, which were initially calculated using farthest point sampling in the Green’s embedding, i.e. using bilinear distances. For efficient nearest neighbor queries we use k-d trees with the “sliding mid-point rule” [52], which adapts to the low intrinsic dimensionality of the data. In summary, the current point-to-point correspondences $C_1$ are inferred from $G$ by:

$$C_1 = \{ (p, \rho_M(p)) \mid p \in S_M \} \cup \{ (\rho_N(q), q) \mid q \in S_N \}$$

(28)

7.4. Iterative alignment

We refine the alignment by alternating between solving for the alignment $G$ using the current point-to-point constraints and calculating novel point-to-point constraints $C_1$ from the current alignment. From the second iteration on solving for $G$ is over-constrained and no further regularization is required. $G$ is then defined as the minimum of:

$$E^C_{C_0} [B, t, \bar{t}] + E^\mathcal{D}_{D_0}[B, t] + E^\mathcal{D}_B[B, t, \bar{t}]$$

(29)

In our experiments we use 20 iterations, which was enough to converge. The effect of iterating is shown in Fig. 7 and the entire alignment procedure is shown in Algorithm 1.

8. Evaluation

An important application of our method is matching non-isometric shapes, which we evaluate on the Shrec dataset [53]. The dataset contains several shape classes, in which we create maps and evaluate their quality using the dense inter-class correspondences from [11] as ground truth.
Algorithm 1 Matching two shapes.

1: **Input**: shapes \( M, N \); initial point-to-point \( C_0 \) and functional \( D_0 \) constraints; \( \alpha \in [0, 1] \)
2: \( S_M/S_N \leftarrow \text{FARthest_POINT_SAMPLING}(M/N, 200) \)
3: \( B, t \leftarrow \text{solve Eq. (27) for an initial alignment} \) using \( C_0, D_0, \alpha, \epsilon = 10^{-6} \)
4: for \( i = 1 \ldots 20 \) do
5: \( C_1 \leftarrow \text{Matches}(B, t, S_M, S_N) \)
6: \( B, t \leftarrow \text{solve Eq. (29) for a novel alignment} \) using \( C_0, D_0, C_1, \alpha \)
7: return Matches\((B, t, M, N)\)

8.1. Initial results

The first results in Figs. 1, 5 and 7 show that good results can be obtained from few sparse point-to-point correspondences. As discussed in Section 4 and shown in Fig. 5, the well-defined distances on the Green’s functions result in a better alignment than distances on delta-distributions. The novel embedding gives smoother maps already in the first iteration, which are further smoothed by the iterative ICP-like alignment. Figs. 5 and 7 show the effect of iterative alignment depicting maps without iterating and after 20 iterations.

8.2. Solving with point-to-point constraints

We compare our method to functional maps (FM), which align delta-distributions, and Blended Intrinsic Maps (BIM), which is a state-of-the-art method for automatic shape matching and does not use predefined constraints. More precisely, when matching with functional maps, we use Eq. (9) to iteratively align the delta-distributions and extract correspondences via nearest neighbor search. We omit the additional rigid ICP alignment described in [15] as it requires isometric shapes and in principle can be applied to our method as well. In each class of the Shrec dataset we build between 30 random maps. Then we build the point-to-point constraints by geodesic farthest point sampling on the source shape and mapping these points onto the target with the ground truth map. To evaluate the maps we equally distributed 200 points on the source, mapped them onto the target and measured the deviation from the ground-truth. Fig. 8a shows the results of the three methods with either three or six point-to-point correspondences and Fig. 8b gives details for three classes. Independent of the number of constraints our method outperforms the functional maps method.

The number of eigenvectors used for the calculations influences the results. Fig. 8c shows a steady improvement as the number of eigenvectors is increased (six point-to-point constraints, \( \alpha = 0 \)). In principle maps are not restricted when represented with Green’s functions. For every map \( T \) there is a Green’s alignment \( G \) and for a finite set of constraints there is an infinite number of possible maps. Yet in our case there are two additional requirements for the solution, namely the alignment must be represented with \( k \) eigenvectors and for \( \alpha > 0 \) the solution must fulfill the conformity conditions. These two requirements can be seen as the degrees-of-freedom of the optimization, as without these requirements any (continuous) map is a valid solution. The degrees-of-freedom are reduced by either decreasing the number of eigenvectors \( k \) or by increasing the weight \( \alpha \) of the conformity constraints. The influence of the degrees-of-freedom on the optimization is shown in Fig. 8d. For three initial point-to-point constraints results improve when degrees-of-freedom are reduced, while for six initial point-to-point constraints the results improve when degrees-of-freedom are increased.

8.3. Solving with functional constraints

We labeled three regions on a representative shape of the classes ‘birds’ and ‘fourlegged’ and transfer these labels onto all shapes of the same class using the ground truth correspondences [11]. Each label consists of four colors, so that we can build four functional constraints from the indicator functions. Fig. 9 shows the labels on the bird class as well as the matching results. Here we intentionally have chosen an example where BIM fails to demonstrate the usefulness of predefined constraints. Fig. 10 depicts a quantitative evaluation of the matching process and further results are depicted in the additional material.

Our method differs from the FM method in two aspects, namely in the novel embedding and in the addition of conformity constraints. We also considered a third method, that uses dual delta-distributions (to constrain and extract point-to-point correspondences) and conformal functional constraints. This third method differs from each of the other two formulations in only one aspect. The results show that both the novel embedding and the conformity constraints improve the results.

While a variety of different combinations of embeddings and constraints are possible only few achieved good results in our experiments. For example, the embedding with dual-delta distributions \((F^D_{0} = \delta_{1})\) does not work well with functional constraints without conformality \((|F| = \delta)\). The reasons might be that the first is a constraint on \( F \), while the second is a constraint on \( F^D \) and mixing constraints on \( F \) and \( F^D \) does not work well (see Table 1).

8.4. Point clouds and topological changes

Due to its underlying simplicity, our method is rather general. It works with point clouds (Fig. 11a) and complicated topologies (Fig. 11b). Fig. 11c shows an example of matching in the presence of severe topological differences, where meshes were sewed together at self-intersections. Apparently our method can be seen as a global intrinsic alignment. Where the matching succeeded, most errors were localized around areas where the topology has changed.

8.5. Isometric matching

The embedding with Green’s functions is most useful when there are few known constraints. If the functional constraints already determine the functional map, there is little difference between embedding with Green’s functions and delta-distributions. For example, when matching isometric shapes the heat kernel signatures and wave kernel signatures [36,37] provide a magnitude
8.6. Blending shapes and limitations

A good demonstration of the quality of the correspondences is their utilization to linearly blend the source triangulation onto the target shape, which results a novel triangulation of the target improved by applying rigid ICP on the delta-distributions as described in [15]. This was not done here as our focus is on near-isometries and as it improves both methods in the same way.
shape. This new triangulation depends on the correspondences and shows their quality. Typically the following errors occur: (1) If the correspondence map is not surjective, i.e. not all target vertices have a corresponding source vertex, then these vertices are not contained in the novel triangulation at all (e.g. see missing vertices on the teddy ear). (2) If neighboring source vertices are mapped differently, then this leads to intersecting edges, outstanding long edges, and edges that are not located on the target shape.

Fig. 13 shows such a blending using our results. While in principle our results are of good quality, there are at least two reasons for the occurring misalignments. One is that Green’s functions are very similar in proximity of thin extrusions (e.g. horse legs) and another is that only a local but not a global optimization was used (e.g. teddy arm). Our results can be further improved using a method for affine point cloud alignment such as Coherent Point Drift [55], which is also shown in Fig. 13 and in much more detail in the additional material.

9. Concluding remarks

We expanded on the understanding of the intrinsic alignment of shapes by reformulating the alignment in an euclidean space with a well-defined metric. Our novel embedding preserves the important advantages of the functional framework, namely that it can be aligned with a linear map, that it is invariant to shape deformations preserving the intrinsic metric and that it can incorporate functional constraints. Using the Green’s embedding has proven to be especially useful for matching non-isometric shapes, where typically only a few correspondences are known. Additionally to the best of our knowledge, we are the first to include conformality as functional constraints into the matching process. Our evaluation shows that to match non-isometric shapes the novel embedding is superior to the previously utilized delta-distributions. Due to its simplicity and generality our method works on shapes of higher genus, on point clouds and to some degree even in the presence of severe changes in the topology and the inner metric. It therefore might as well serve as a basis for further development of techniques for shape matching.

Appendix

Proposition 1a: Injectivity of $g^M : \mathcal{M} \rightarrow \mathcal{L}^2(\mathcal{M})$, $p \mapsto g^M_p$: Let $p, q \in \mathcal{M}$ with $g^M_p = g^M_q$ and $f \in \mathcal{L}^2(\mathcal{M})$, then $0 = \langle \Delta f, (g^M_p - g^M_q) \rangle = f(p) - f(q) \Rightarrow p = q$.

Proposition 1b: Let $\beta \in \mathcal{L}^2(\mathcal{M})$ be the coefficients of a linear combination $h(y) = \int_q \beta(q)(\phi_i + g_i(y)) dq \in \mathcal{L}^2(\mathcal{M})$. then

$$\langle \phi_i, h \rangle = \langle \phi_i, \beta \rangle \begin{cases} \sqrt{\lambda} & i = 1 \\ 1/\lambda_i & i \geq 1 \end{cases}$$

$$\langle \phi_i, h \rangle = \int_q \beta(q)(\phi_i + g_i) dq = \int_q \beta(q) dq = \sqrt{\lambda}(\phi_i, \beta)$$

"i $\geq 2" TOOL$: $\lambda_i(\phi_i, h) = (\Delta \phi_i, h) = (\phi_i, \Delta h) = \int q \beta(q) \Delta(\phi_i + g_i) dq \int q \beta(q) \Pi \delta_q dq$ = $\langle \phi_i, \int q \beta(q) \Pi dq \rangle = \langle \phi_i, \Pi(\beta) \rangle = \langle \phi_i, \beta \rangle$

"Independent": Let $h$ be a linear combination as defined above with $h(y) = 0$, then also $\beta = 0$.

"Spans $\mathcal{L}^2(\mathcal{M})\"$: Because the $\phi_i$ form a basis of $\mathcal{L}^2(\mathcal{M})$ it is sufficient to represent the $\phi_i$ with linear combinations of the $g^M_p$. Choosing $\beta = \phi_i$ results in $h = \mu \phi_i$ for some $\mu \neq 0$. 
Supplementary material association with this article can be found, in the online version, at 10.1016/j.cag.2017.06.004.

References